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On four-point functions of $\frac{1}{2}$-BPS operators in general dimensions

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Abstract: We study four-point correlation functions of $\frac{1}{2}$-BPS operators of arbitrary weight for all dimensions $d = 3, 4, 5, 6$ where superconformal theories exist. Using harmonic superspace techniques, we derive the superconformal Ward identities for these correlators and present them in a universal form. We then solve these identities, employing Jack polynomial expansions. We show that the general solution is parameterized by a set of arbitrary two-variable functions, with the exception of the case $d = 4$, where in addition functions of a single variable appear. We also discuss the operator product expansion using recent results on conformal partial wave amplitudes in arbitrary dimension.

Keywords: Field Theories in Higher Dimensions, AdS-CFT and dS-CFT Correspondence, Extended Supersymmetry.

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1. Introduction

The AdS/CFT correspondence has stimulated an enormous activity in the study of superconformal theories in the past few years. The most typical example is the $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions whose conjectured holographic dual is the type-IIB supergravity (string) theory on $AdS_5 \times S^5$ [1]. Another case of interest is the superconformal six-dimensional theory of the $(2,0)$ self-dual tensor multiplet. It is believed to describe the world-volume fluctuations of the M-theory five-branes on $AdS_7 \times S^4$ [2]. Three-dimensional $\mathcal{N} = 8$ superconformal theories have been considered in relation to anti-de Sitter black holes of gauged SO(8) supergravity on $AdS_4 \times S^7$ [3].

A considerable part of this recent activity has been concentrated on the so-called $\frac{1}{2}$-BPS states. These are very special representations of the underlying superconformal symmetry group of the above theories, which are annihilated by half of the supercharges. Thus, the superconformal kinematics fixes their conformal weight (dimension) at its free theory value. Such states (or the corresponding composite operators in field theory) are called “short” or “protected”. Translated in terms of correlation functions, this means that the two-point functions of $\frac{1}{2}$-BPS operators do not receive any quantum corrections. An even stronger result is that their three-point functions are also protected. Non-trivial dynamics starts appearing at the level of the four-point functions. There one sees the rich spectrum of the OPE of $\frac{1}{2}$-BPS operators, which contains protected as well as unprotected (“long multiplet”) states. Superconformal kinematics puts strong restrictions on the general form of such correlators and on their OPE content, which have been successfully tested against numerous results in perturbative field theory and in AdS supergravity.

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In this paper we present a systematic study of the superconformal properties of four-point functions of $\frac{1}{2}$-BPS operators in all cases of interest. Namely, we derive and solve the constraints (or superconformal Ward identities) following from superconformal symmetry in spaces of dimension $d = 3, 4, 5, 6$. The corresponding superconformal algebras are (P)SU$(2, 2|\mathcal{N})$ with $\mathcal{N} = 2, 4$ in $d = 4$, OSp$(8^*|2\mathcal{N})$ with $\mathcal{N} = 1, 2$ in $d = 6$, OSp$(8|4, \mathbb{R})$ in $d = 3$, as well as the exceptional algebra $F(4)$ in $d = 5$.

An early attempt at such a study was undertaken in [4], where the simplest case of $\frac{1}{2}$-BPS operators of weight two in $d = 4$ $\mathcal{N} = 2$ superconformal theory was considered and the corresponding superconformal Ward identities were derived using harmonic/analytic superspace techniques [5, 6]. This result was generalized in [7] to the four-point functions of stress-tensor multiplets (or $\frac{1}{2}$-BPS operators of weight two) in $d = 4$ $\mathcal{N} = 4$ super-Yang-Mills theory. There the general solution to the Ward identities was found, making use of the full crossing symmetry of the amplitude. It was shown that the solution is parameterized by a function of a single conformally invariant variable and by another, arbitrary function of two variables. A dynamical argument based on the field-theoretic insertion procedure [8] lead to the conclusion in [7] that the single-variable function does not receive quantum corrections (a phenomenon called "partial non-renormalization"), in excellent agreement with results from $AdS_5 \times S^5$ supergravity [9] and from perturbative field theory [10]. Later on, in [11] the same Ward identities were derived, in a component field approach, and solved without using crossing symmetry. There it was also shown that the single-variable part of the solution (which, without crossing symmetry, involves two independent single-variable functions) admits an OPE (or conformal partial wave) expansion over a restricted class of superconformal representations, the so-called short and semishort multiplets. Four-point correlators of $\frac{1}{2}$-BPS operators of weight $k > 2$ in the $d = 4$ $\mathcal{N} = 4$ theory were studied in [12, 13, 14]. In particular, in [12], using analytic superspace techniques it was shown that for arbitrary k the amplitude is parameterized by $\frac{1}{2}k(k-1)$ functions of two variables and by k independent single-variable functions. The latter were shown to be expanded over protected superconformal states only.[1] Partial non-renormalization (i.e., the absence of quantum corrections to the single-variable functions) for $k = 3, 4$ was confirmed both perturbatively and in AdS supergravity in [13, 14]. An alternative explanation of this phenomenon was given in [12], where it was linked to the protectedness of the three-point functions of short and semishort multiplets.

The implications of superconformal symmetry for four-point functions in six spacetime dimensions were studied in [16]. There the Ward identities for the four-point function of composite operators of weight two, made out of $(2,0)$ self-dual tensor multiplets, were derived and solved, using crossing symmetry. In contrast with the four-dimensional case, it was found here that the solution could be parameterized by a single two-variable function. Again, these general predictions were compared with success to explicit results from $AdS_7 \times S^4$ supergravity.

In the present paper we are able to extend the result of [16] to the $d = 6$ case[2] of

[1] See also [15] where similar results are obtained in a component field approach.

[2] See also some comments about $d = 6$ in [17].

arbitrary weight k and without crossing symmetry, as well as to the cases $d = 3, 5$. In the process of solving the superconformal Ward identities we obtain a set of arbitrary two-variable functions, as well as another set of restricted functions. Unlike the case $d = 4$, the latter can be absorbed into suitable redefinitions of the former. This is possible because in $d \neq 4$ both types of functions are acted upon by a differential operator with a non-trivial kernel. It is precisely this kernel which is responsible for the absorption of the restricted sector of the solution. In the $d = 4$ case this operator is the identity and the absorption is impossible.

The method we use in this paper for deriving the Ward identities is based on the appropriate versions of harmonic superspace for each type of superconformal symmetry. The defining property of the $\frac{1}{2}$-BPS operators, realized in $d = 3, 4, 5, 6$ superspace, is that they depend only on half of the Grassmann (odd) variables. Such superfields should not be confused with the familiar $d = 4$ $\mathcal{N} = 1$ chiral superfields. In the chiral case the half is chosen with regard to the Lorentz group (left- or right-handed chirality), and this only makes sense in $d = 4$. The $\frac{1}{2}$-BPS superfields depend on a different half of the odd variables obtained by projecting their R symmetry index rather than the Lorentz one. Thus, $\frac{1}{2}$-BPS superfields can be defined independently of the space-time dimension. Such superfields are called "Grassmann analytic" [18, 5, 6]. Grassmann analyticity can be achieved without loss of manifest R symmetry if one introduces internal symmetry (also called "harmonic") variables. The resulting Grassmann analytic harmonic superspace (or its version called "analytic superspace" [6, 12]) is the "native" superspace for the realization of $\frac{1}{2}$-BPS representations of the various superconformal algebras in $d = 3, 4, 5, 6$ (see [19, 20] for a review on BPS states in harmonic/analytic superspaces).

Another technique we use in this paper is the expansion of the four-point amplitude in a basis of two-variable symmetric polynomials, the so-called Jack polynomials [21]. The Ward identities constraining these four-point functions take a very simple and universal form for all dimensions d, upon making a judicious choice of variables parametrizing the conformal and R symmetry invariants. However, their solution is far from trivial, because the functions satisfying the Ward identities are symmetric separately in the conformal and R symmetric invariant variables. Without this symmetry the solutions are not difficult to find, but the Jack polynomial expansion is indispensable when it comes to taking the symmetry requirements into account. Such polynomials have been used before to find, in general dimensions, conformal partial wave amplitudes representing contributions to any scalar four-point correlation function of an operator, and its descendants, in the OPE of two of the operators in the four-point function [22]. Since the conformal group $\mathrm{SO}(2, d)$ is non-compact, the conformal partial wave amplitudes are infinite-order polynomials in the conformal invariants. For four dimensions the resulting expansion is directly equivalent to one in terms of Schur polynomials, which were used in [12] to find $\mathcal{N} = 4$ four-point functions compatible with superconformal symmetry. We use the expansion of [22] in terms of Jack polynomials to perform conformal partial wave analysis of the $k = 2$ solution.

Let us now briefly describe the content of the paper. In section 2 we discuss the four-point invariant variables associated with the d-dimensional conformal group $\mathrm{SO}(2, d)$, as well as with the $\mathrm{SO}(n)$ R symmetry group of the corresponding superconformal theories.

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These symmetries can be realized on light-like vectors of $SO(2, d)$ (see, e.g., [23]) or on complex "light cone" (or "null") vectors of $SO(n)$.[3] They serve as the coordinates of space-time or of the internal (harmonic) space, respectively. We then introduce a frame fixing procedure, which consists in using all the available conformal/internal symmetry in order to reduce the coordinates to just two independent invariant variables in the space-time sector and to two (for R symmetry $SO(n)$, $n > 3$) or one (for R symmetry $SO(3)$) variable in the internal sector.[4] We also discuss the general conformal and R symmetry properties of two-, three- and four-point functions of the primary scalar states of the $\frac{1}{2}$-BPS multiplets. We write these amplitudes as polynomial expansions in the internal invariant variables with coefficients which are functions of the space-time invariants.

In section 3 we apply the same frame fixing procedure[5] to the two-, three- and four-point invariants in the Grassmann analytic superspaces suitable for the description of $\frac{1}{2}$-BPS superfields. We choose a superconformal frame involving only the conformal and internal invariant variables, as well as the Grassmann analytic variables at one of the four points. These variables undergo linear residual supersymmetry transformations, which allow us to easily find the completion of the even variables to superconformal invariants. We show that these invariants become singular when one of the internal variables approaches one of the space-time variables.

In section 4 we use these results to supersymmetrize the four-point amplitudes. The crucial point here is the requirement that the amplitude be free from singularities. This is precisely the origin of the superconformal Ward identities. The frame fixing procedure guarantees that the constraints we find are necessary and sufficient conditions for superconformal invariance. We present the Ward identities in a universal form suitable for any space-time dimension and any R symmetry group. In this section we also solve the Ward identities in the $d = 4$ case, thus reproducing the known results.

In section 5 we discuss the Ward identities for theories in any dimension d and with R symmetry group $SO(3)$ (these include the cases $d = 6$ $\mathcal{N} = (1, 0)$, $d = 5$ and $d = 4$ $\mathcal{N} = 2$). We need to solve two partial differential equations involving the $k + 1$ coefficient functions of the amplitude of weight k. We start by finding the general solution in the simplest case of weight $k = 1$ in terms of Bessel functions of the first and second kind, depending on whether the dimension of the spacetime is odd or even, respectively. Subsequently we show that the $k = 2$ case can be reduced to the $k = 1$ one. This is done by a suitable redefinition of the coefficients of the amplitude, for which we employ an operator identity. This identity is also used to show that we may solve for the general k case in a similar fashion. We find that the solution is parameterized by $k - 1$ unrestricted two-variable functions along with

[3] The use of such variables probably goes back to the beginning of the 20th century, see [24]. They have been employed, in a very explicit form, in the studies of conformal field theories in the 1970s (see [25, 26] and references therein). In the context of four-point functions of $\frac{1}{2}$-BPS operators null vectors have been used in [13, 14], and recently in [15].

[4] Such variables have already been used in [7, 11, 12] for solving the superconformal Ward identities. Here we give them another, very simple interpretation as, e.g., the light-cone coordinates of two-dimensional space-time.

[5] This generalizes the method proposed in [4]. A similar procedure has been used before in [27] in the context of $d = 4$ $\mathcal{N} = 1$ supersymmetry.

the restricted solution from the $k = 1$ case. An important point is that the restricted part of the solution for each $k \geq 2$, coming from the $k = 1$ reduction, may be completely absorbed into the two-variable part via a certain redefinition of the latter, but only for $d \neq 4$. This is shown here for $d = 6$ but holds more generally, as we see in section 6 where Jack polynomials are employed to find solutions.

In section 6 we solve the Ward identities for theories with R symmetry SO(n), $n \geq 3$ (these include the cases $d = 3$ $\mathcal{N} = 8$, $d = 4$ $\mathcal{N} = 4$ and $d = 6$ $\mathcal{N} = (2,0)$). We use expansions in terms of Jack polynomials to solve for the simplest $k = 1$ case, showing how solutions for the $k = 1$ case for SO(3) may be unified in terms of these. We also prove and generalize certain claims made in section 5 pertaining to arbitrary dimension. Finally, we solve for the $k = 2$ and general k cases using these and other results based on Jack polynomial expansion. The solution is parameterized by $\frac{1}{2}k(k-1)$ unrestricted two-variable functions and k restricted ones which, for $d \neq 4$, may be absorbed into a redefinition of the two-variable functions.

In section 7 we use results from [22] to expand the $k = 2$ solutions to the Ward identities for SO(n) in terms of conformal partial wave amplitudes. This is nontrivial for the unrestricted two-variable part of the solution. Still, we are able to conclude that the conformal partial wave expansion of this part of the solution corresponds to long multiplets in the OPE, whose primary state is a scalar. It is also shown here how one of the restricted solutions, even though it can be absorbed into a redefinition of a two-variable function, corresponds to shortened multiplets in the OPE, whose primary state is again a scalar and all operators appearing have twist $d - 2$. Also in this section, the corresponding results of [11] for $d = 4$ $\mathcal{N} = 4$ are recovered.

Generalizing this approach to conformal partial wave expansion for other k appears very non-trivial — at least when using the method described here. We hope that the basis independent form of the solutions to the Ward identities for $d \neq 4$, might aid such expansion more generally. We feel that the issue is related to making an appropriate choice of basis with which to expand the solution, so as to extract harmonic polynomials in the R symmetry invariants and conformal partial waves.

Note added. After the first version of this paper was submitted to the e-archive, the paper [39] appeared in which the case $d = 6$, $k = 2$ is treated by a different method. In section 6.2.1 we show that our results for this case and those of [39] are equivalent.

2. Conformal and R symmetries of the four-point amplitudes

The subject of our study are the four-point correlation functions of the so-called $\frac{1}{2}$-BPS operators. The $\frac{1}{2}$-BPS states, also called “short supermultiplets”, are special representations of the superconformal algebras in $d = 3, 4, 5, 6$. The symmetry group of the even sector of these subalgebras is the product of the d-dimensional conformal group SO($2, d$) and the R symmetry group. In all the cases of interest the R symmetry group is locally isomorphic to an orthogonal group SO($n + 2$) with $n = 1, 3, 4$ or 6. Postponing the discussion of the superconformal properties to section 3, here we concentrate on the bosonic symmetries of these amplitudes.

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2.1 Conformal cross-ratios and related variables

Dirac proposed a natural realization of the representations of the conformal group, the so-called "light ray" realization (see, e.g., [23]). The idea is to introduce real light-like (or null) vector variables X^m, $m = 0, 1, \dots, d+1$,

$$X \cdot X = (X^0)^2 - (X^1)^2 - \cdots - (X^d)^2 + (X^{d+1})^2 = 0\,, \tag{2.1}$$

on which the conformal group $\mathrm{SO}(2,d)$ acts linearly. The representations of the conformal group can be realized on fields $\phi(X)$ on the light cone, homogeneous of degree ℓ, $\phi(\rho X) = \rho^\ell \phi(X)$. The algebraic condition (2.1) and the homogeneity condition effectively eliminate two degree of freedom, so we are left with d independent variables. A convenient choice for the d coordinates of space-time is

$$X^m = \left(x^\mu, \frac{1}{2}(1+x^2), \frac{1}{2}(1-x^2)\right)\,, \qquad \mu = 0, 1, \dots, d-1\,. \tag{2.2}$$

Here $x^2 = (x^0)^2 - (x^1)^2 - \cdots - (x^{d-1})^2$ and x^μ is a vector of the Lorentz group $\mathrm{SO}(1, d-1)$ on which the conformal group $\mathrm{SO}(2,d)$ acts *non-linearly.*

One of our main goals is to construct invariants. In the case of the conformal group this is quite obvious. Since the vectors X^m are light-like, we cannot form an $\mathrm{SO}(2,d)$ invariant with a single vector. With two such vectors we can obtain a covariant by taking their scalar product $X_1 \cdot X_2$. Here covariance means that the homogeneity condition requires that each vector transform with its own scaling factor, $X_1 \cdot X_2 \to \rho_1 \rho_2 X_1 \cdot X_2$. Then it is clear that invariants can be constructed by using four null vectors and forming the so-called "conformal cross-ratios"

$$u = \frac{X_1 \cdot X_2\, X_3 \cdot X_4}{X_1 \cdot X_3\, X_2 \cdot X_4}\,, \qquad v = \frac{X_1 \cdot X_4\, X_2 \cdot X_3}{X_1 \cdot X_3\, X_2 \cdot X_4}\,. \tag{2.3}$$

In terms of the unconstrained d-vectors $x^\mu_{1,2,3,4}$ these cross-ratios read

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,, \tag{2.4}$$

where $x_{ij}^2 = (x_i - x_j)^2$.

In what follows we shall work in a special conformal frame, commonly used in studies of conformal theories. With the help of translations, conformal boosts, dilatations and part of the Lorentz group $\mathrm{SO}(1, d-1)$, we can fix the frame[6]

$$x_2^\mu = (1, 0, \dots, 0)\,, \qquad x_3^\mu = (x_4^\mu)^{-1} = 0\,, \tag{2.5}$$

where $(x_4^\mu)^{-1} = x_4^\mu / x_4^2$. This choice breaks $\mathrm{SO}(1, d-1)$ down to $\mathrm{SO}(d-1)$. We can specify the frame even further by using the residual rotational symmetry $\mathrm{SO}(d-1)$, so that the vector x_1 has only two independent components:

$$x_1^\mu = \left(1 + \frac{1}{2}(z+z'), \frac{1}{2}(z-z'), 0, \dots, 0\right)\,, \tag{2.6}$$

[6]In this paper, in order to avoid space-time singularities we always assume that $x_i \neq x_j$.

or, equivalently,

$$x_{12}^{\mu} = x_1^{\mu} - x_2^{\mu} = \left(\frac{1}{2}(z+z'), \frac{1}{2}(z-z'), 0, \ldots, 0\right). \tag{2.7}$$

Thus, in the frame (2.5), (2.7) the cross-ratios (2.4) become

$$u = \frac{zz'}{(1+z)(1+z')}, \qquad v = \frac{1}{(1+z)(1+z')}. \tag{2.8}$$

In fact, such variables are most natural and are widely used in two-dimensional conformal field theory.[7] Indeed, let us replace the vector coordinates x^{μ} of the two-dimensional space by the light-cone coordinates

$$\zeta^{\pm} = x^0 \pm x^1 \;\Leftrightarrow\; x^{\mu} = \left(\frac{1}{2}(\zeta^+ + \zeta^-), \frac{1}{2}(\zeta^+ - \zeta^-)\right), \qquad \zeta^+\zeta^- = x^2. \tag{2.9}$$

The two-dimensional Lorentz group SO(1,1) acts by rescaling of $\zeta^{\pm}$, so we need not square the coordinate differences $\zeta^{\pm}_{ij}$ to obtain Lorentz invariant cross-ratios. Thus, we can form the cross-ratios

$$\begin{aligned} z &= \frac{\zeta^+_{12}\zeta^+_{34}}{\zeta^+_{14}\zeta^+_{23}}, \qquad & z' &= \frac{\zeta^-_{12}\zeta^-_{34}}{\zeta^-_{14}\zeta^-_{23}}; \\ 1+z &= \frac{\zeta^+_{13}\zeta^+_{24}}{\zeta^+_{14}\zeta^+_{23}}, \qquad & 1+z' &= \frac{\zeta^-_{13}\zeta^-_{24}}{\zeta^-_{14}\zeta^-_{23}}. \end{aligned} \tag{2.10}$$

Substituting (2.10) in the expressions for u, v (2.4), we obtain precisely (2.8). It is then natural to extend the definition (2.8) to any space-time dimension.

It should be stressed that the change of variables (2.8) is not bijective. Indeed, solving eqs. (2.8) for z, z' we find

$$z = \frac{1}{2v}\left(1-u-v\pm\sqrt{\Delta}\right), \qquad z' = \frac{1}{2v}\left(1-u-v\mp\sqrt{\Delta}\right), \tag{2.11}$$

where

$$\Delta = (1-u-v)^2 - 4uv = -4\frac{\epsilon\mathbf{x}^2}{((x^0)^2+\epsilon\mathbf{x}^2)^2}. \tag{2.12}$$

To obtain (2.12) we have used the conformal frame (2.5) with $x_1^{\mu} = (x^0, \mathbf{x})$, and with $\epsilon = -1$ in the Minkowski case and $\epsilon = +1$ in the euclidean case. This explains why the variables z, z' are real in the former case and complex in the latter. The choice of sign in (2.11) is conventional and should not affect any physical quantity. This follows from the definition of z, z' (2.8), which is symmetric under the exchange $z \leftrightarrow z'$. Therefore, all the results we are going to obtain should have this symmetry.

We note that the equation $z = z'$ (i.e., $\Delta = 0$) defines the "border line" between the euclidean and minkowskian domains in the conformal cross-ratios plane. The physical correlation functions should be well defined in the entire minkowskian (euclidean) domain, including the border. We shall come back to this point later on.

[7] E.S. is grateful to I. Todorov for this remark.

The relevance of such variables to the superconformal Ward identities was first pointed out in [7]. The variables ξ, η used there are related to z, z' as follows:

$$z \to \xi\,, \qquad z' \to \frac{1}{\eta}\,. \tag{2.13}$$

They are the standard variables for solving the integrability condition for the superconformal Ward identities. They are also commonly used in the literature on multiloop integrals (see, e.g., [28]). Later on in ref. [11] it was shown that a more convenient choice of variables allows one to directly solve the Ward identities without considering their integrability condition. The variables z', x of [11] are related to our z, z' as follows:

$$z \to \frac{z'}{1-z'}\,, \qquad z' \to \frac{x}{1-x}\,. \tag{2.14}$$

Finally, in ref. [12] it was pointed out that such variables arise in yet another way, by diagonalizing the coordinate matrix (see section 3.3).

2.2 R symmetry invariant variables

Besides the conformal group, the amplitudes we are discussing in this paper have another symmetry, the compact R symmetry group which is locally isomorphic to an orthogonal group SO$(n+2)$ with $n = 1, 3, 4$ or 6. Its representations can be realized in terms of internal variables similar to the space-time variables of the conformal group.

2.2.1 The cases SO(5), SO(6) and SO(8)

In these cases the $\frac{1}{2}$-BPS multiplets have primary states in symmetric traceless tensor representations of SO(n). For example, the scalars of the $d = 4$ $\mathcal{N} = 4$ SYM multiplet form an SO(6) vector, while the primary state of the stress-tensor multiplet is in the **20'** of SU(4) $\sim$ SO(6) (a rank two tensor of SO(6)). It is natural to associate such representations to *complex null vectors* of SO$(n+2)$. The idea is very much like that of Dirac's light ray realization of the conformal group. One introduces null vector variables $Y^{\mathbf{i}}$, $\mathbf{i} = 1, \ldots, n+2$,

$$Y \cdot Y = (Y^1)^2 + \cdots + (Y^{n+2})^2 = 0\,, \tag{2.15}$$

on which the group SO$(n+2)$ acts linearly. This condition, together with a homogeneity requirement similar to that in the conformal case, reduces the SO$(n+2)$ vector Y to an independent SO(n) vector, e.g.,

$$Y^{\mathbf{i}} = \left(y^i, \frac{1}{2}(1-\vec{y}^2), \frac{i}{2}(1+\vec{y}^2)\right), \qquad i = 1, \ldots, n\,, \tag{2.16}$$

on which SO$(n+2)$ acts *non-linearly*. Note that the SO$(n+2)$ null vector $Y^{\mathbf{i}}$ must be complex, and so is the unconstrained SO(n) vector y^i.

Given four sets of such internal coordinates we can construct R symmetry invariants in close analogy with the conformal case:

$$U = \frac{Y_1 \cdot Y_2\, Y_3 \cdot Y_4}{Y_1 \cdot Y_3\, Y_2 \cdot Y_4}\,, \qquad V = \frac{Y_1 \cdot Y_4\, Y_2 \cdot Y_3}{Y_1 \cdot Y_3\, Y_2 \cdot Y_4}\,. \tag{2.17}$$

In terms of the independent n-vectors $y^i_{1,2,3,4}$ these internal cross-ratios read

$$U = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}\,, \qquad V = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}\,. \tag{2.18}$$

Using part of the SO($n+2$) symmetry[8] we can fix the frame (cf. (2.5))

$$y^i_2 = (1, 0, \dots, 0)\,, \qquad y^i_3 = (y^i_4)^{-1} = 0\,. \tag{2.19}$$

This breaks SO($n+2$) down to SO($n-1$). Then, using this residual rotational symmetry, we can further fix the frame

$$\vec{y}_{12} = \left(\frac{1}{2}(w+w'), \frac{1}{2}(w-w'), 0, \dots, 0\right), \tag{2.20}$$

in which there remain only two independent complex variables w, w'. In terms of these new variables the invariant cross-ratios read (cf. (2.8))

$$U = \frac{ww'}{(1+w)(1+w')}\,, \qquad V = \frac{1}{(1+w)(1+w')}\,. \tag{2.21}$$

2.2.2 The case SO(3)

This case is special since the elementary $\frac{1}{2}$-BPS multiplets are related to the fundamental (spinor) rather than the vector representation of SU(2) $\sim$ SO(3). For example, the scalars of the $d=4$ $\mathcal{N}=2$ matter multiplet (hypermultiplet) belong to an SU(2) doublet. Then the internal variables should form a spinor $Y^{\mathbf{a}}$, $\mathbf{a}=1,2$, of SU(2) (or of its complexification SL$(2,\mathbb{C})$). Once again, we consider functions of Y homogeneous of degree ℓ, $\phi(\rho Y) = \rho^\ell \phi(Y)$, where ρ is complex.[9] This allows us to choose a frame in which $Y^{\mathbf{a}}$ has only one independent component,

$$Y^{\mathbf{a}} = (y, 1)\,. \tag{2.22}$$

Further, given two points we can form the SL$(2,\mathbb{C})$ invariant contraction

$$Y_1 \cdot Y_2 \equiv Y_1^{\mathbf{a}} \epsilon_{\mathbf{ab}} Y_2^{\mathbf{b}} = Y_1^{\mathbf{1}} Y_2^{\mathbf{2}} - Y_1^{\mathbf{2}} Y_2^{\mathbf{1}} = y_1 - y_2 \equiv y_{12}\,. \tag{2.23}$$

Given four points and taking into account the cyclic identity

$$y_{12} y_{34} - y_{13} y_{24} + y_{14} y_{23} = 0\,, \tag{2.24}$$

we can construct a single independent invariant variable, e.g.,

$$w = \frac{Y_1 \cdot Y_2\, Y_3 \cdot Y_4}{Y_1 \cdot Y_4\, Y_2 \cdot Y_3} = \frac{y_{12} y_{34}}{y_{14} y_{23}}\,. \tag{2.25}$$

We may say that this case is in a sense the analog of the $d=2$ conformal one.

Finally, we can use the SL$(2,\mathbb{C})$ symmetry to fix the frame

$$y_1 = 1 + w\,, \qquad y_2 = 1\,, \qquad y_3 = y_4^{-1} = 0\,. \tag{2.26}$$

Note that this choice completely breaks down the R symmetry, the remaining independent variable w being invariant.

[8] Strictly speaking, since the vectors $\vec{y}$ are complex, we should use the complexification of SO(n). However, in what follows we shall not make this distinction.

[9] The internal variables we use here are similar, although not identical, to the "harmonic" variables of [5]. The latter form an SU(2) matrix and are homogeneous with a U(1) factor.

2.3 Two-, three- and four-point functions

The most important property of the $\frac{1}{2}$-BPS operators is that their two-, three- and four-point functions are completely determined by the lowest (primary) component in their superfield expansion (see section 3). This lowest component is in turn strongly restricted by conformal and R symmetry covariance.

2.3.1 Two- and three-point functions

The best-known example of $\frac{1}{2}$-BPS states are the composite operators made out of the $d=4$ $\mathcal{N}=4$ super-Yang-Mills on-shell field-strength multiplet. The lowest component of this multiplet is a set of six real scalars $\phi^{\mathbf{i}}(x)$, $\mathbf{i}=1,\ldots,6$ in the vector representation of SO(6). Introducing the SO(6) null vector (2.16) we can define the scalar field

$$\phi(x,y) = i\sqrt{2}\phi^{\mathbf{i}}(x)Y^{\mathbf{i}} = \frac{i}{\sqrt{2}}\left[(\phi^5+i\phi^6)+2\phi^i y^i-(\phi_5-i\phi_6)\vec{y}^2\right]. \tag{2.27}$$

This field, regarded as a function of the null vector $Y^{\mathbf{i}}$, transforms with unit weight under the projective action of the R symmetry group,

$$\phi(x,\rho Y) = \rho\phi(x,Y). \tag{2.28}$$

Notice that the dependence on the internal variables y^i is polynomial.[10]

In this notation the two-point function of the free scalars

$$\langle\phi^{\mathbf{i}}(x_1)\phi^{\mathbf{j}}(x_2)\rangle = \frac{\delta^{\mathbf{ij}}}{x_{12}^2} \tag{2.29}$$

becomes

$$\langle\phi(x_1,y_1)\phi(x_2,y_2)\rangle = \frac{Y_1\cdot Y_2}{x_{12}^2} = \frac{y_{12}^2}{x_{12}^2}. \tag{2.30}$$

The $\mathcal{N}=4$ $\frac{1}{2}$-BPS primary states of weight k are realized as gauge-invariant composite operators

$$\mathcal{O}^k(x,y) = \mathrm{Tr}(\phi(x,y))^k = \mathrm{Tr}(\phi^{\mathbf{i}_1}\cdots\phi^{\mathbf{i}_k})Y^{\mathbf{i}_1}\cdots Y^{\mathbf{i}_k}, \tag{2.31}$$

transforming in the symmetric traceless representation of SO(6) (or the $[0,k,0]$ of SU(4)),

$$\mathcal{O}^k(x,\rho Y) = \rho^k\mathcal{O}^k(x,Y). \tag{2.32}$$

These operators have conformal weight equal to their R weight k, in the free as well as in the interacting field theory. The most remarkable feature of the BPS operators is that their conformal dimension is "protected", i.e. does not receive any quantum corrections. The two-point function of such operators is just the k-th power of (2.30),

$$\langle\mathcal{O}^k(x_1,y_1)\mathcal{O}^k(x_2,y_2)\rangle = \left(\frac{Y_1\cdot Y_2}{x_{12}^2}\right)^k = \left(\frac{y_{12}^2}{x_{12}^2}\right)^k. \tag{2.33}$$

[10] The null vector Y is complex, therefore the projected field $\phi(x,y)$ cannot be real. Nevertheless, in the harmonic [5] (or analytic [12]) superspace approach one can define a combination of complex conjugation and of a particular reflection on the internal manifold, under which $\phi(x,y)$ goes into itself and thus the coefficients in (2.27) can be made real.

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Exactly as in ordinary conformal theory, where the three-point functions of scalars are obtained by multiplying two-point functions, we can construct the three-point function of $\frac{1}{2}$-BPS operators of weights k_1, k_2, k_3 as follows:

$$\langle \mathcal{O}^{k_1}(x_1,y_1)\mathcal{O}^{k_2}(x_2,y_2)\mathcal{O}^{k_3}(x_3,y_3)\rangle = \\ = C(g)\left(\frac{y_{12}^2}{x_{12}^2}\right)^{\frac{k_1+k_2-k_3}{2}}\left(\frac{y_{13}^2}{x_{13}^2}\right)^{\frac{k_1+k_3-k_2}{2}}\left(\frac{y_{23}^2}{x_{23}^2}\right)^{\frac{k_2+k_3-k_1}{2}}, \tag{2.34}$$

where $C(g)$ is a normalization constant, in general depending on the gauge coupling g. Remarkably, for $\frac{1}{2}$-BPS operators not only their weights k, but also the normalization $C(g)$ is "protected" [29].

Another, even simpler example of a $\frac{1}{2}$-BPS multiplet is the elementary $d = 4$ $\mathcal{N} = 2$ matter multiplet (hypermultiplet). It can be obtained as a submultiplet of the $\mathcal{N} = 4$ SYM multiplet under the reduction $\mathcal{N} = 4 \to \mathcal{N} = 2$. The lowest (primary) component of the hypermultiplet is an SU(2) doublet of complex scalars $\phi_{\mathbf{a}}(x)$ whose two-point function is

$$\langle \phi_{\mathbf{a}}(x_1)\bar{\phi}^{\mathbf{b}}(x_2)\rangle = \frac{\delta_{\mathbf{a}}{}^{\mathbf{b}}}{x_{12}^2}\,. \tag{2.35}$$

Introducing the internal variables from section 2.2.2, we can write down the doublet as a polynomial of degree one, $\phi(x,y) = \phi_{\mathbf{a}}(x)Y^{\mathbf{a}} = \phi_2(x) + y\phi_1(x)$. Analogously, the conjugate fields become $\widetilde{\phi}(x,y) = \bar{\phi}^{\mathbf{a}}\epsilon_{\mathbf{ab}}Y^{\mathbf{b}} = \bar{\phi}^1(x) - y\bar{\phi}^2(x)$, where we have used the same variables $Y^{\mathbf{a}}$ and not their conjugates. In this notation the two-point function (2.35) takes the form

$$\langle \phi(x_1,y_1)\widetilde{\phi}(x_2,y_2)\rangle = \frac{Y_1\cdot Y_2}{x_{12}^2} = \frac{y_{12}}{x_{12}^2}\,. \tag{2.36}$$

We can also define $\mathcal{N} = 2$ $\frac{1}{2}$-BPS composite operators with primary states $\mathcal{O}^k = \mathrm{Tr}(\phi(x,y))^k$ of weight k (or isospin $2k$) whose two-point function is the k-th power of (2.36):

$$\langle \mathcal{O}^k(x_1,y_1)\widetilde{\mathcal{O}}^k(x_2,y_2)\rangle = \frac{y_{12}^k}{x_{12}^{2k}}\,. \tag{2.37}$$

The internal variables in (2.37) form a polynomial of degree k. Three-point functions are defined by analogy with (2.34).

The $d = 4$ cases discussed above can immediately be generalized to $\frac{1}{2}$-BPS states in $d = 3, 5, 6$. The d-dimensional free scalar fields have canonical conformal dimension

$$\varepsilon = \frac{d}{2} - 1\,. \tag{2.38}$$

Accordingly, the two-point function of the scalars from the elementary on-shell supermultiplet becomes

$$\langle \phi^{\mathbf{i}}(x_1)\phi^{\mathbf{j}}(x_2)\rangle = \frac{\delta^{\mathbf{ij}}}{(x_{12}^2)^{\varepsilon}}\,, \tag{2.39}$$

where $\mathbf{i}, \mathbf{j}$ are indices of the corresponding R symmetry group. Multiplying by the appropriate vector (or spinor) internal variables Y we obtain the two-point functions of elementary and composite states:

$$\langle \phi(x_1,y_1)\phi(x_2,y_2)\rangle = \frac{Y_1\cdot Y_2}{(x_{12}^2)^{\varepsilon}} = \frac{y_{12}^2}{(x_{12}^2)^{\varepsilon}}\,. \tag{2.40}$$

2.3.2 Four-point functions

Let us first discuss the case $d = 4$ $\mathcal{N} = 4$. The four-point correlator of the primary states of $\frac{1}{2}$-BPS operators of weight k is obtained by connecting each point to the other three points with k elementary two-point functions ("propagators") in all possible ways. This gives rise to $\frac{1}{2}(k+1)(k+2)$ propagator structures, each having the required conformal and R symmetry properties. The general amplitude is then given by a linear combination of these structures with invariant coefficients depending on the conformal cross-ratios u, v:

$$\begin{aligned}\langle \mathcal{O}^k(x_1,y_1)\mathcal{O}^k(x_2,y_2)\mathcal{O}^k(x_3,y_3)\mathcal{O}^k(x_4,y_4)\rangle^{\mathcal{N}=4} &= \\ = \sum_{0\le m+n\le k} a_{mn}(u,v)\left(\frac{y_{12}^2y_{34}^2}{x_{12}^2x_{34}^2}\right)^{k-m-n}\left(\frac{y_{13}^2y_{24}^2}{x_{13}^2x_{24}^2}\right)^{m}\left(\frac{y_{14}^2y_{23}^2}{x_{14}^2x_{23}^2}\right)^{n}.\end{aligned} \tag{2.41}$$

The dependence on the internal variables y is polynomial, which reflects the fact that we are dealing with finite-dimensional representations of $SU(4) \sim SO(6)$. The number of terms $\frac{1}{2}(k+1)(k+2)$ in this sum is also the number of $SU(4)$ representations occurring in the tensor product of two representations $[0,k,0]$. The polynomial of degree k of two variables in (2.41) provides a "propagator basis" for such correlators. It is related, although in a non-trivial way, to the OPE basis where the expansion goes over $SU(4)$ representations — see section 7.

The weights of the four operators in (2.41) do not have to be the same. Considering a set of four different weights $k_1 \ge k_2 \ge k_3 \ge k_4$, we can generalize (2.41) to the following factorized form:

$$\begin{aligned}\langle \mathcal{O}^{k_1}(x_1,y_1)\mathcal{O}^{k_2}(x_1,y_1)\mathcal{O}^{k_3}(x_3,y_3)\mathcal{O}^{k_4}(x_4,y_4)\rangle &= \\ = \langle \mathcal{O}^{k_1'}(x_1,y_1)\mathcal{O}^{k_2'}(x_2,y_2)\mathcal{O}^{k_3'}(x_3,y_3)\rangle &\times \\ \times \langle \mathcal{O}^{k_4}(x_1,y_1)\mathcal{O}^{k_4}(x_2,y_2)\mathcal{O}^{k_4}(x_3,y_3)\mathcal{O}^{k_4}(x_4,y_4)\rangle &,\end{aligned} \tag{2.42}$$

where $k_i' = k_i - k_4$. As we shall see later on, this generalization has no effect on the superconformal Ward identities, the latter concern only the coefficients of the correlator of weight k_4 in (2.42).

We remark that if the four operators $\mathcal{O}^k$ are considered identical, the amplitude is invariant under point permutations ("crossing symmetry"). This symmetry imposes certain relations among the coefficient functions a_{mn}. In the present paper we do not consider the implications of crossing symmetry.

It is convenient to rewrite the amplitude (2.41) by pulling out a single propagator factor of weight k,

$$\begin{aligned}\langle \mathcal{O}^k(x_1,y_1)\mathcal{O}^k(x_2,y_2)\mathcal{O}^k(x_3,y_3)\mathcal{O}^k(x_4,y_4)\rangle^{\mathcal{N}=4} &= \\ = \left(\frac{y_{12}^2y_{34}^2}{x_{12}^2x_{34}^2}\right)^{k}\sum_{0\le m+n\le k} a_{mn}(u,v)\left(\frac{V}{v}\right)^{n}\left(\frac{u}{U}\right)^{m+n}.\end{aligned} \tag{2.43}$$

This factor gives the amplitude the required conformal and R weight, while the sum is an invariant function of the cross-ratios (2.3), (2.18).

Further, we can go to the conformal and R symmetry frames (2.5), (2.7), (2.19), (2.20) and rewrite the amplitude as a function of the invariant variables z, z' and w, w':

$$\lim_{x_4,y_4\to\infty}\left(\frac{x_4^2}{y_4^2}\right)^k \langle \mathcal{O}^k(x_1,y_1)\mathcal{O}^k(x_2,y_2)\mathcal{O}^k(x_3,y_3)\mathcal{O}^k(x_4,y_4)\rangle^{\mathcal{N}=4} = \left(\frac{y_{12}^2}{x_{12}^2}\right)^k G_k(z,z';w,w')\,, \tag{2.44}$$

where

$$G_k(z,z';w,w') = \sum_{0\le m+n\le k} a_{mn}(z,z')\left(\frac{(1+w)(1+w')}{(1+z)(1+z')}\right)^m\left(\frac{zz'}{ww'}\right)^{m+n}\,. \tag{2.45}$$

The coefficient functions $a_{mn}(z,z')$ should satisfy the additional condition

$$a_{mn}(z,z') = a_{mn}(z',z)\,. \tag{2.46}$$

Indeed, initially these coefficients were functions of the cross-ratios, $a_{mn}(u,v)$. As pointed out earlier, in the change of variables (2.11) from u, v to z, z' there is a sign ambiguity. The choice of this sign is conventional and should not affect any physical quantity. The same applies to the internal variables. So, we define

$$G_k(z,z';w,w') = G_k(z',z;w,w') = G_k(z,z';w',w)\,. \tag{2.47}$$

This requirement amounts to extending the superconformal transformations by global exchange symmetries [12].

The four-point amplitude is even simpler in the case $d=4$ $\mathcal{N}=2$. Instead of (2.41) we have

$$\langle \mathcal{O}^k(x_1,y_1)\mathcal{O}^k(x_2,y_2)\mathcal{O}^k(x_3,y_3)\mathcal{O}^k(x_4,y_4)\rangle^{\mathcal{N}=2} = \sum_{n=0}^{k} a_n(z,z')\left(\frac{y_{12}y_{34}}{x_{12}^2x_{34}^2}\right)^{k-n}\left(\frac{y_{14}y_{23}}{x_{14}^2x_{23}^2}\right)^n. \tag{2.48}$$

The appearance of only two distinct propagator structures in (2.48) follows from the cyclic identity (2.24). The sum has $k+1$ terms which is also the number of SU(2) representations occurring in the tensor product of two representations of isospin $k/2$ each. Further, we can pull out a propagator factor, e.g., $y_{12}^k y_{34}^k/x_{12}^{2k}x_{34}^{2k}$, which gives the correlator the required conformal and SU(2) weight. The rest of it then becomes an invariant polynomial of degree k in the single variable

$$\frac{y_{14}y_{23}}{y_{12}y_{34}}\frac{x_{12}^2x_{34}^2}{x_{14}^2x_{23}^2} \to \frac{zz'}{w} \tag{2.49}$$

in the frame (2.5), (2.7), (2.26). So, in this frame we have

$$\lim_{x_4,y_4\to\infty}\left(\frac{x_4^2}{y_4}\right)^k \langle \mathcal{O}^k(x_1,y_1)\mathcal{O}^k(x_2,y_2)\mathcal{O}^k(x_3,y_3)\mathcal{O}^k(x_4,y_4)\rangle^{\mathcal{N}=2} = \left(\frac{y_{12}}{x_{12}^2}\right)^k G_k(z,z';w)\,, \tag{2.50}$$

where

$$G_k(z,z';w) = \sum_{n=0}^{k} a_n(z,z')\left(\frac{zz'}{w}\right)^n\,. \tag{2.51}$$

The generalization of the above to other dimensions is obvious as we just need to use the corresponding propagators. Thus, the four-point amplitude (2.44) becomes

$$\lim_{x_4,y_4\to\infty}\left(\frac{x_4^{2\varepsilon}}{y_4^2}\right)^k \langle \mathcal{O}^k(x_1,y_1)\mathcal{O}^k(x_2,y_2)\mathcal{O}^k(x_3,y_3)\mathcal{O}^k(x_4,y_4)\rangle^{(\varepsilon)} = \left(\frac{y_{12}^2}{x_{12}^{2\varepsilon}}\right)^k G_k^{(\varepsilon)}(z,z';w,w')\,, \tag{2.52}$$

where

$$G_k^{(\varepsilon)}(z,z';w,w') = \sum_{0\le m+n\le k} a_{mn}(z,z')\left(\frac{(1+w)(1+w')}{(1+z)^\varepsilon(1+z')^\varepsilon}\right)^m \left(\frac{z^\varepsilon z'^\varepsilon}{ww'}\right)^{m+n}\,, \tag{2.53}$$

and similarly for (2.51),

$$G_k^{(\varepsilon)}(z,z';w) = \sum_{m=0}^{k} a_m(z,z')\left(\frac{z^\varepsilon z'^\varepsilon}{w}\right)^n\,. \tag{2.54}$$

Note that (2.54) can be obtained from (2.53) by taking the limit $w'\to\infty$ and by identifying

$$a_m(z,z') = \sum_{n=m}^{k} C_n^m \frac{a_{n0}(z,z')}{(1+z)^{\varepsilon n}(1+z')^{\varepsilon n}}\,, \tag{2.55}$$

where C_n^m are the binomial coefficients, $(1+1)^n = \sum_m C_n^m$.

For the purposes of solving the superconformal Ward identities (sections 5, 6) and for application to conformal partial wave expansion (section 7) it is convenient to make the following change of variables:

$$\chi = \frac{z}{1+z}\,, \qquad \chi' = \frac{z'}{1+z'}\,, \qquad \alpha = \frac{1+w}{w}\,, \qquad \alpha' = \frac{1+w'}{w'}\,. \tag{2.56}$$

In terms of these we have

$$G_k^{(\varepsilon)}(\chi,\chi';\alpha,\alpha') = \sum_{0\le m+n\le k} a_{mn}(\chi,\chi')\,\frac{(\chi\chi')^{\varepsilon(m+n)}}{[(1-\chi)(1-\chi')]^{\varepsilon n}}\,(\alpha\alpha')^m[(1-\alpha)(1-\alpha')]^n \tag{2.57}$$

and, by setting $\alpha'=1$,

$$G_k^{(\varepsilon)}(\chi,\chi';\alpha) = \sum_{m=0}^{k} a_m(\chi,\chi')\,(\chi\chi')^{\varepsilon m}\,\alpha^m\,, \tag{2.58}$$

where now $a_m(\chi,\chi') = a_{m0}(\chi,\chi')$.

3. Superconformal symmetry and $\frac{1}{2}$-BPS states

3.1 Superconformal covariants and invariants of the $\frac{1}{2}$-BPS type

The four-point amplitudes considered in section 2.3.2 transform as covariants of the conformal and R symmetry group of weight k at each point. Our main task in this section is to complete them to full superconformal covariants.

The crucial property of the two-, three- and four-point superconformal (co)invariants of the $\frac{1}{2}$-BPS type is that their dependence on the Grassmann variables is uniquely fixed by superconformal symmetry, given the lowest (i.e., $\theta = 0$) component in their expansion. This is quite obvious from the following counting argument. A four-point Grassmann-analytic (co)invariant depends on four sets of odd variables $\theta_{1,2,3,4}$, each having half of the components of a Lorentz and R symmetry spinor. On the other hand, the superconformal algebra has two sets of odd parameters, one for Q-supersymmetry and another for S-supersymmetry, each being a full spinor. Disregarding any possible singularities, we conclude that these parameters can be used to choose a frame in which all $\theta_{1,2,3,4} = 0$. Inversely, "undoing" this frame we obtain the unique superconformal completion of a conformal and R symmetry four-point (co)invariant of this type. Another way of putting this is to say that there exist no nilpotent four-point superconformal invariants of the Grassmann analytic type. Obviously, the same applies to (co)invariants with two or three points.

The simple counting argument above seems to imply that the four-point correlators of $\frac{1}{2}$-BPS operators are entirely determined by the properties of their lowest component, i.e., just by conformal invariance and R symmetry, and that superconformal symmetry is implemented by straightforward completion of the conformal (co)invariants to superconformal ones. This is not exactly true, and the reason is the singular nature of the superconformal transformations needed to remove all the θs. As to the potential space-time singularities, we may postulate that we keep all four points $x_{1,2,3,4}$ apart, so that no such singularities occur. However, we are not allowed to treat the harmonic (R symmetry) singularities in the same way. The point is that the harmonic functions are *globally defined analytic functions* on the harmonic space. The latter is a complex compact coset of the R symmetry group. This property of *harmonic analyticity* guarantees that the harmonic dependence is just polynomial. In other words, we want do deal with finite-dimensional unitary representations of the compact R symmetry group which can be written down in the form of harmonic polynomials.

3.2 Two- and three-point superconformal covariants

The analytic (polynomial) dependence of the harmonic variables may indeed be lost when we try to remove all the θs. A simple example illustrates this phenomenon very well — the two-point function of the elementary $d = 4$ $\mathcal{N} = 2$ on-shell superfield, the hypermultiplet. In section 2.3.1 we have found its form (2.36) as a function of the space-time variables x_{12} and as a *polynomial* in the harmonic variables y_{12}. The question now is to find its completion to a full superconformal covariant and to make sure that the polynomial dependence on the harmonic variables persists.

The hypermultiplet can be described [5] by a $\frac{1}{2}$-BPS (or Grassmann analytic) superfield which, as its name suggests, depends only on half of the Grassmann variables $\theta^{\alpha}{}_{\mathbf{a}}$ and $\bar{\theta}^{\alpha'}{}_{\mathbf{a}}$. Here α, α' are Lorentz spinor indices and $\mathbf{a}$ is a spinor index of the R symmetry group SU(2). The half suitable for Grassmann analyticity is obtained by a harmonic projection of the type described in section 2.2.2,

$$\theta^{\alpha}_{+} = \theta^{\alpha}{}_{\mathbf{a}} Y^{\mathbf{a}} , \qquad \bar{\theta}^{\alpha'}_{+} = \bar{\theta}^{\alpha'}{}_{\mathbf{a}} Y^{\mathbf{a}} , \tag{3.1}$$

so that they only carry a U(1) (or GL$(1, \mathbb{C})$ after complexification) weight.

We do not need to study the full superconformal group for the supersymmetrization of the two-point function (2.36). It is well known that the conformal two-point functions are determined essentially by Poincaré invariance together with dilatations (no need to use proper conformal invariance). Similarly, their Grassmann-analytic superconformal extensions are completely fixed by Poincaré (or Q-) supersymmetry, R symmetry and dilatations (no need to use conformal, or S-supersymmetry). Adapting the counting argument from the beginning of this section to Q-supersymmetry alone, we see that a two-point function of the Grassmann analytic type depends on as many odd variables $(\theta^{\alpha}_{+})_{1,2}$, $(\bar\theta^{\alpha}_{+})_{1,2}$ as the number of Q-supersymmetry parameters, so it is indeed completely determined by Q-supersymmetry.

Let us consider the left-handed part of Q-supersymmetry with parameters $\epsilon^{\alpha}{}_{\mathbf{a}}$. The transformations of the coordinates of the Grassmann analytic superspace have the following form:

$$\delta x^{\alpha\alpha'} = \epsilon^{\alpha}_{-}\bar\theta^{\alpha'}_{+}\,, \qquad \delta\theta^{\alpha}_{+} = \epsilon^{\alpha}{}_{\mathbf{a}}Y^{\mathbf{a}} = \epsilon^{\alpha}_{+} + y\epsilon^{\alpha}_{-}\,, \qquad \delta\bar\theta^{\alpha'}_{+} = \delta y = 0\,, \tag{3.2}$$

where $x^{\alpha\alpha'} = x^{\mu}(\sigma_{\mu})^{\alpha\alpha'}$ and the Q-supersymmetry parameters ϵ^{α} are projected with the internal variable Y, in accordance with (3.1). These transformations allow us to fix a special frame similar to the conformal and R symmetry frames discussed in section 2,

$$\text{left-handed Q frame:} \qquad \theta^{\alpha}_{2+} = 0\,. \tag{3.3}$$

In this frame we still have the residual Q-supersymmetry with parameters

$$\epsilon^{\alpha}_{+} = -y_2\epsilon^{\alpha}_{-}\,, \tag{3.4}$$

under which the remaining θ_{1+} undergoes the transformation

$$\delta_Q\theta^{\alpha}_{1+} = y_{12}\epsilon^{\alpha}_{-}\,. \tag{3.5}$$

It is then clear that if we wish to go a step further and eliminate θ_{1+} as well, we would create a harmonic singularity of the type y_{12}^{-1}, and this would not be compatible with the requirement of harmonic polynomial dependence (i.e., with harmonic analyticity). So, we must keep θ_{1+} in order to have control on the harmonic singularities.

The extension of x_{12} to a Q-supersymmetry invariant is very easy to find[11] in the left-handed Q frame (3.3), together with the analogous right-handed frame $\bar\theta^{\alpha'}_{2+} = 0$:

$$\check{x}^{\alpha\alpha'}_{12} = x^{\alpha\alpha'}_{12} - y_{12}^{-1}\theta^{\alpha}_{1+}\bar\theta^{\alpha'}_{1+}\,. \tag{3.6}$$

When the two harmonic points coincide, $y_{12} = 0$, the supersymmetric extension (3.6) is clearly singular. What matters however is the singularity of the complete supersymmetrized two-point function (2.36),

$$\langle\phi(x_1,\theta_1,\bar\theta_1,y_1)\widetilde{\phi}(x_2,0,0,y_2)\rangle = \frac{y_{12}}{\check{x}^2_{12}}\,. \tag{3.7}$$

[11] A similar construction for two- and three-point functions in $d=4$ $\mathcal{N}=1$ superspace has been considered in [31], and in [32] for $d=3,6$.

Expanding in $\theta\bar{\theta}$, we see that the harmonic numerator suppresses the singularity of the denominator at the level $\theta\bar{\theta}$. At the top level $(\theta\bar{\theta})^2$ the space-time factor is $\Box_1 x_{12}^{-2} = 0$ (recall that we neglect space-time singularities). Thus, the supersymmetric two-point function (3.7) is *free from harmonic singularities.*[12]

The full dependence on both θ_1 and θ_2 in (3.7) can be restored by "undoing" the frame fixing, i.e. by performing the inverse Q-supersymmetry transformation which lead to (3.3) and to its right-handed analog. This does not change the behaviour of the two-point function at the coincident point $y_{12} = 0$.

The supersymmetrization of the two-point functions (2.37) of $\frac{1}{2}$-BPS operators of weight k is achieved in exactly the same way, by just replacing x by $\check{x}$:

$$\langle \mathcal{O}^k(x_1, \theta_1, \bar{\theta}_1, y_1) \widetilde{\mathcal{O}}^k(x_2, 0, 0, y_2) \rangle = \frac{y_{12}^k}{\check{x}_{12}^{2k}} . \tag{3.8}$$

We have obtained the two-point functions (3.7), (3.8) using Q-supersymmetry alone. However, from the counting argument above we know that Q-supersymmetry is sufficient to guarantee the uniqueness of the two-point functions of the $\frac{1}{2}$-BPS type. This implies that (3.7), (3.8) automatically have the required transformation properties under the full superconformal symmetry.

The other elementary $\frac{1}{2}$-BPS multiplet in $d = 4$, the on-shell $\mathcal{N} = 4$ SYM multiplet, is treated similarly [30]. Here the harmonic variables form an $SO(4) \sim SU(2) \times SU(2)$ vector $y^{aa'}$ and the odd variables of the Grassmann analytic superspace $\theta^{\alpha a'}$, $\bar{\theta}^{a\alpha'}$ carry Lorentz and SU(2) spinor indices of both types. Now (3.6) becomes

$$\check{x}_{12}^{\alpha\alpha'} = x_{12}^{\alpha\alpha'} - \theta_1^{\alpha a'} (y_{12})^{-1}_{a'a} \bar{\theta}_1^{a\alpha'} \tag{3.9}$$

and the supersymmetrization of the two-point function (2.30) is

$$\langle \phi(x_1, \theta_1, \bar{\theta}_1, y_1) \phi(x_2, 0, 0, y_2) \rangle = \frac{y_{12}^2}{\check{x}_{12}^2} . \tag{3.10}$$

Expanding in $\theta_1\bar{\theta}_1$, we see that the harmonic numerator suppresses the singularity of the denominator up to the level $(\theta_1\bar{\theta}_1)^2$. At the next levels the space-time factor contains $\Box_1 x_{12}^{-2} = 0$. Thus, the supersymmetric two-point function (3.10) is free from harmonic singularities.

Three-point superconformal correlators of $\frac{1}{2}$-BPS states are obtained by simply replacing $x_{ij}^2 \to \check{x}_{ij}^2$ in (2.34). Once more, counting the number of θs versus the number of Q- and S-supersymmetry parameters, we see that this supersymmetric extension is unique. Moreover, it is also free from harmonic singularities, since each propagator factor in it has this property.

Finally, the generalization of the above construction to $d = 3, 5, 6$ is straightforward and is left to the reader.[13]

[12] In fact, the top level in the expansion of (3.7) is a contact term, $y_{12}^{-1}\delta(x_{12})(\theta_1\bar{\theta}_1)^2$. This is needed for the two-point function to become the solution of the Green's function equation for the hypermultiplet [5].

[13] We just mention that the harmonic superspace formulation of the $d = 6$ $(2,0)$ self-dual tensor multiplet was given in [33]. The properties of the two- and three-point functions of $\frac{1}{2}$-BPS operators for general dimensions have been discussed in [34, 35], also in a harmonic superspace framework.

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3.3 Four-point superconformal invariants

The four-point amplitude (2.43) is a conformal and R symmetry covariant of weight k at each point. Actually, the necessary weight is carried by the propagator factor, the coefficient functions $a_{mn}(u,v)$ are invariant. The supersymmetrization of this amplitudes goes in two steps. Firstly, we replace $x_{ij}^2 \to \hat{x}_{ij}^2$ in the propagator factor, as explained in section 3.2. This step does not create any harmonic singularities since the propagators are free from them. Secondly, we have to complete the space-time u, v and harmonic U, V cross-ratios to full superconformal invariants.

Instead of the cross-ratios, it is easier to supersymmetrize the alternative variables z, z' and w, w'. A systematic procedure for constructing such four-point invariants has been proposed in [12].[14] However, as explained in section 4, for the purpose of deriving the superconformal Ward identities we need only the linearized (lowest order in the θ expansion) invariants. This is easily done by examining the linearized supersymmetry transformations and finding the appropriate $\theta\theta$ terms which compensate the variations of z, z' and w, w'. We find it very useful to generalize the conformal and R symmetry frames of section 2 to the superconformal case.

Let us now look into the details, case by case.

3.3.1 $d = 6$, $\mathcal{N} = (2,0)$ and $\mathcal{N} = (1,0)$

Here we are considering the superalgebra $\mathrm{OSp}(8^*|2\mathcal{N})$ with $\mathcal{N} = 1, 2$, the supersymmetric extension of the $d = 6$ conformal group is $\mathrm{SO}(2,6)$. The corresponding space-time coordinates form an $\mathrm{SO}(1,5)$ vector x^μ. It is convenient to rewrite x^μ as a bispinor (matrix) by contracting it with gamma matrices,

$$x^{\alpha\beta} = -x^{\beta\alpha} = x^\mu(\Gamma_\mu)^{\alpha\beta}\,. \tag{3.11}$$

We recall that $d = 6$ supersymmetry is chiral, so only gamma matrices of one chirality are used.[15] The spinor indices α, β belong to the fundamental representation of $SU^*(4) \sim \mathrm{SO}(1,5)$. The R symmetry group in the case $\mathcal{N} = (2,0)$ is $\mathrm{USp}(4) \sim \mathrm{SO}(5)$, so according to section 2.2 we need an $\mathrm{SO}(3)$ vector coordinate y^i. The latter can be cast in matrix form,

$$y^{ab} = y^{ba} = y^i(\gamma^i)^{ab} \tag{3.12}$$

where the spinor indices a, b belong to the fundamental representation of $\mathrm{SU}(2)$ and γ^i are the $\mathrm{SO}(3)$ gamma matrices, i.e., the three symmetric Pauli matrices. The odd coordinates of the Grassmann analytic superspace $\theta^{\alpha a}$, corresponding to $\frac{1}{2}$-BPS multiplets, carry both space-time ($SU^*(4)$) and internal ($\mathrm{SU}(2)$) spinor indices. Note that the full $d = 6$ $\mathcal{N} = (2,0)$ superspace contains twice this number of odd variables, forming a spinor of $\mathrm{USp}(4)$.

[14] In appendix A we apply a version of this procedure to one particular case, $d = 4$ $\mathcal{N} = 4$, in order to answer a question raised in section 4.

[15] The $d = 6$ case is a good illustration of how inadequate the term "Chiral Primary Operators", commonly used to denote $\frac{1}{2}$-BPS operators, is. In $d = 6$ every supermultiplet is chiral. The characteristic feature of the $\frac{1}{2}$-BPS superfields is that they depend on half of the odd coordinates of this chiral superspace.

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In this paper we work in a frame where $x_2^\mu = (1, 0, \ldots, 0)$ and $y_2^i = (1, 0, 0)$. The matrix form of these fixed vectors is obtained by making a convenient choice of the relevant gamma matrices in the space-time and internal sectors:

$$\Gamma_0 = \begin{pmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \qquad \gamma_1 = \sigma_0 = \delta^{ab}. \tag{3.13}$$

In this frame the residual Lorentz symmetry is SO(5) and the R symmetry is further broken from SO(3) down to SO(2), so the position of the indices a, b is now irrelevant.

In what follows we need the Q- and S-supersymmetry transformations of the superspace coordinates. In section 2 we fixed conformal and R symmetry four-point frames in which the independent even variables are the coordinates at point 1. In the supersymmetric version of these frames we can eliminate all the odd variables but θ_1:

$$\theta_2^{\alpha a} = \theta_3^{\alpha a} = (x_4^{-1})_{\alpha'\alpha}\theta_4^{\alpha a} = 0. \tag{3.14}$$

This is done by using 3/4 of the Q- and S-supersymmetry transformations. The remaining 1/4 supersymmetry should be compatible with this choice. It is not hard to see that the residual supersymmetry transformations of the independent coordinates $x = x_1 - x_2$, $y = y_1 - y_2$ and $\theta = \theta_1$ should have the form[16]

$$\begin{aligned} \delta x^{\alpha\beta} &= -\epsilon^{\alpha b}\theta^{\beta b} - \theta^{\alpha b}\epsilon^{\beta b} \\ \delta y^{ab} &= \epsilon^{\alpha a}(\tilde{\Gamma}_0)_{\alpha\beta}\theta^{\beta b} + \theta^{\alpha a}(\tilde{\Gamma}_0)_{\alpha\beta}\epsilon^{\beta b} \\ \delta\theta^{\alpha a} &= y^{ab}\epsilon^{\alpha b} - x^{\alpha\beta}(\tilde{\Gamma}_0)_{\beta\gamma}\epsilon^{\gamma a}, \end{aligned} \tag{3.15}$$

where $(\tilde{\Gamma}_0)_{\alpha\beta} = -\Gamma_0^{\alpha\beta}$, $\Gamma_0\tilde{\Gamma}_0 = \mathbb{I}$. Indeed, from (3.13) it is clear that $\delta\theta_2 = \sigma_0\epsilon - \Gamma_0\tilde{\Gamma}_0\epsilon = 0$. In addition, since by fixing the conformal and R symmetry frames we have already used most of the bosonic symmetry (translations, conformal boosts, dilatations, part of SO(1, 5) and SO(5)), only the residual SO(5) $\times$ SO(2) should appear in the commutator of two supersymmetry transformations (3.15). This explains the linear form of the transformations (3.15) and even fixes the coefficients, up to a simultaneous rescaling of θ and ϵ.

We now recall that the space-time and internal space frames can be further specified as shown in (2.7), (2.19). The choice

$$\Gamma_1 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad \gamma_2 = \sigma_3, \tag{3.16}$$

makes the matrices x and y diagonal:

$$x^\alpha{}_\beta \equiv x^{\alpha\gamma}(\tilde{\Gamma}_0)_{\gamma\beta} = \begin{pmatrix} z & & & \\ & z' & & \\ & & z & \\ & & & z' \end{pmatrix}, \qquad y^{ab} = \begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}. \tag{3.17}$$

[16] The complete superconformal transformations can be derived in the supermatrix approach of [12]. After fixing the corresponding frame they reduce to (3.15). The point we want to make here is that working directly in this frame allows us to obtain all that we need in just a few elementary steps.

This choice casts the supersymmetry transformation of θ into a very suggestive form:

$$
\begin{aligned}
\delta\theta^1 &= \begin{pmatrix} w-z & & & \\ & w-z' & & \\ & & w-z & \\ & & & w-z' \end{pmatrix} \epsilon^1 + (\epsilon\theta\theta)\,; \\
\delta\theta^2 &= \begin{pmatrix} w'-z & & & \\ & w'-z' & & \\ & & w'-z & \\ & & & w'-z' \end{pmatrix} \epsilon^2 + (\epsilon\theta\theta)\,,
\end{aligned} \tag{3.18}
$$

where $\theta^{1,2}$ are the two projections of the R symmetry doublet and $(\epsilon\theta\theta)$ denotes non-linear terms to be discussed below. We see that each component of θ^α is shifted by the corresponding parameter ϵ^α multiplied by the difference of an internal (w, w') and a space-time (z, z') invariant variables. Were it not for these differences which might vanish, we could proceed to a further frame fixing where all the θs are eliminated (recall the counting argument from the beginning of section 3). However, the corresponding supersymmetry transformations are clearly singular when, e.g., $w \to z$, etc. The requirement of absence of such singularities is the origin of the superconformal Ward identities, as explained in section 4.

Next, we give the transformations of the even variables:

$$
\begin{aligned}
\delta z &= \epsilon^{1a}\theta^{3a} + \theta^{1a}\epsilon^{3a} \\
\delta z' &= \epsilon^{2a}\theta^{4a} + \theta^{2a}\epsilon^{4a} \\
\delta w &= -2\epsilon^{\alpha 1}(\tilde{\Gamma}_0)_{\alpha\beta}\theta^{\beta 1} \\
\delta w' &= -2\epsilon^{\alpha 2}(\tilde{\Gamma}_0)_{\alpha\beta}\theta^{\beta 2}\,.
\end{aligned} \tag{3.19}
$$

We stress that in (3.18) we only see the linearized part of the supersymmetry transformations of the θs. Indeed, the particular choice (3.17) is not invariant under the transformations (3.15). In order to restore the diagonal form (3.17) we need to make additional, compensating SO(5) $\times$ SO(2) transformations. For instance, in the space-time sector the SO(5) transformation is given by the matrix $A^\alpha{}_\beta$, $A^\alpha{}_\alpha = 0$ with entries, e.g.,

$$
A^1{}_2 = \frac{1}{z'-z}(\epsilon^{1a}\theta^{4a} + \theta^{1a}\epsilon^{4a})\,, \quad \text{etc.} \tag{3.20}
$$

This does not affect the transformation of z, z' in (3.19), but creates new, non-linear terms of the type $A^\alpha{}_\beta\theta^{\beta a}$ in $\delta\theta^{\alpha a}$ (3.18). The compensating transformation in the internal sector has a similar effect. It is important that these non-linear terms do not create new singularities of the $1/(w-z)$, etc. type (notice, however, the singularity $1/(z'-z)$).

Now we are ready to construct the superconformal completions of the variables z, z', w, w'. Inspecting the linearized transformations (3.18), (3.19), we see that the following expressions:

$$
\begin{aligned}
\hat{z} &= z - \frac{\theta^{11}\theta^{31}}{w-z} - \frac{\theta^{12}\theta^{32}}{w'-z} + O((\theta)^4) \\
\hat{z}' &= z' - \frac{\theta^{21}\theta^{41}}{w-z'} - \frac{\theta^{22}\theta^{42}}{w'-z'} + O((\theta)^4) \\
\hat{w} &= w - 2\frac{\theta^{11}\theta^{31}}{w-z} - 2\frac{\theta^{21}\theta^{41}}{w-z'} + O((\theta)^4) \\
\hat{w}' &= w' - 2\frac{\theta^{12}\theta^{32}}{w'-z} - 2\frac{\theta^{22}\theta^{42}}{w'-z'} + O((\theta)^4)
\end{aligned} \tag{3.21}
$$

are invariant to lowest order in θ,

$$
\delta\hat{z} = \delta\hat{z}' = \delta\hat{w} = \delta\hat{w}' = O(\epsilon(\theta)^3)\,. \tag{3.22}
$$

To obtain the full non-linear invariants we ought to take into account the compensating transformations (3.20), as well as to transform the denominators in (3.21), which leads to higher-order terms. This process terminates after a few steps, but is rather cumbersome. There exists a more efficient method for constructing the full invariants (see [12] and appendix A). However, as explained in section 4, for the purposes of deriving the superconformal Ward identities it is sufficient to know the linearized form (3.21).

The most important feature of the supersymmetric extensions (3.21) is their singular behavior when, e.g., $w \to z$, etc. Our main task in section 4 will be to find the conditions for suppressing these singularities in the expansion of the four-point amplitude. In fact, we already know some particular combinations of the variables (3.21) which should be free form such singularities. Indeed, as shown in section 3.2, the supersymmetrized propagators are not singular. Combining four six-dimensional propagators, we can construct conformal and R symmetry invariants of the cross-ratio type, e.g.,

$$
\frac{y_{12}^2 y_{34}^2}{y_{14}^2 y_{23}^2}\frac{x_{14}^4 x_{23}^4}{x_{12}^4 x_{34}^4} \;\to\; \frac{y_{12}^2}{x_{12}^4} = \frac{ww'}{(zz')^2}\,, \tag{3.23}
$$

where we have used the conformal and R symmetry frames (2.5), (2.19). We have already seen such invariants in the expansion of the amplitude $G_k^{(\varepsilon)}$ (2.53) (here $\varepsilon = 2$). The supersymmetrization of (3.23) can be achieved by replacing $x_{12} \to \check{x}_{12}$, as explained in section 3.2. Alternatively, since the four-point superconformal invariants of the $\frac{1}{2}$-BPS type are unique, this should be equivalent to supersymmetrizing the variables w, w', z, z' in (3.23) according to (3.21):

$$
\frac{y_{12}^2}{\check{x}_{12}^4} = \frac{\hat{w}\hat{w}'}{(\hat{z}\hat{z}')^2}\,. \tag{3.24}
$$

Now, we know from section 3.2 that the left-hand side of (3.24) is free from harmonic singularities, i.e., it is well defined for any value of the "running" variable y_1 (y_2 is fixed in the frame (2.19)). Consequently, the right-hand side of (3.24) should also be well defined for any values of w, w'. In particular, the singularity, e.g., $(w-z)^{-1}$ should be absent. To check this we pick out the singular terms

$$\begin{aligned}\Delta z &= \hat{z} - z = -\frac{\theta^{11}\theta^{31}}{w-z} + \text{reg. terms}\,,\\ \Delta w &= \hat{w} - w = -2\frac{\theta^{11}\theta^{31}}{w-z} + \text{reg. terms}\,,\end{aligned} \tag{3.25}$$

i.e., $\Delta w_{\text{sing}} = 2\Delta z_{\text{sing}}$ in this particular regime. Then the potentially singular factors in the right-hand side of (3.24) combine into the regular term

$$(\Delta z_{\text{sing}}\partial_z + \Delta w_{\text{sing}}\partial_w)\left(\frac{w}{z^2}\right) = \frac{2(z-w)}{z^3}\Delta z_{\text{sing}} = \frac{2}{z^3}\theta^{11}\theta^{31} \tag{3.26}$$

(recall that here we only construct linearized superconformal invariants). The same applies to the other singularities present in (3.21).

Anticipating the discussion of the other dimensions below, we can state that (3.26) generalizes to

$$\Delta w_{\text{sing}} = \varepsilon\Delta z_{\text{sing}} \sim (w-z)^{-1}\,, \qquad (\Delta z_{\text{sing}}\partial_z + \Delta w_{\text{sing}}\partial_w)\left(\frac{w}{z^{\varepsilon}}\right) = \text{regular} \tag{3.27}$$

in the singular regime $w \to z$, and similarly for the other singular limits.

Finally, the case $\mathcal{N} = (1,0)$ is easily obtained by reduction. There the R symmetry group is USp(1) $\sim$ SU(2), so we have a single complex variable $w = y_1 - y_2$ in the internal sector. The reduction is done by suppressing the R symmetry indices a, b and by setting $w' = \infty$ in the equations above. In particular, we find

$$\begin{aligned}\hat{z} &= z - \frac{\theta^1\theta^3}{w-z} + O((\theta)^4)\\ \hat{z}' &= z' - \frac{\theta^2\theta^4}{w-z'} + O((\theta)^4)\\ \hat{w} &= w - 2\frac{\theta^1\theta^3}{w-z} - 2\frac{\theta^2\theta^4}{w-z'} + O((\theta)^4)\,.\end{aligned} \tag{3.28}$$

3.3.2 $d = 5$

Here we are considering the exceptional superalgebra $F(4)$, the unique supersymmetric extension of the $d=5$ conformal group is SO(2,5), with R symmetry USp(1) $\sim$ SU(2). This case can be treated in close analogy with the case $d=6$ $\mathcal{N}=(1,0)$ above, we just need to adapt the supersymmetry transformations (3.15). The reduction from the $d=6$ Lorentz group SO(1,5) to the $d=5$ one SO(1,4) is done using the SO(1,4) invariant symplectic matrix

$$\Omega^{\alpha\beta} = -\Omega_{\alpha\beta} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0\end{pmatrix} = \Gamma_5\,, \qquad \Omega_{\alpha\beta}\Omega^{\beta\gamma} = \delta_\alpha{}^\gamma\,, \tag{3.29}$$

so that the $d=5$ matrix $x^{\alpha\beta}$ is symplectic-traceless, $x^{\alpha\beta}\Omega_{\beta\alpha} = 0$. Then (3.15) becomes

$$\begin{aligned}\delta x^{\alpha\beta} &= -\epsilon^\alpha\theta^\beta - \theta^\alpha\epsilon^\beta - \frac{1}{2}\Omega^{\alpha\beta}\,\epsilon^\gamma\Omega_{\gamma\delta}\theta^\delta\\ \delta y &= \frac{3}{2}\epsilon^\alpha(\tilde{\Gamma}_0)_{\alpha\beta}\theta^\beta\\ \delta\theta^\alpha &= y\epsilon^\alpha - x^{\alpha\beta}(\tilde{\Gamma}_0)_{\beta\gamma}\epsilon^\gamma\,,\end{aligned} \tag{3.30}$$

so that $\delta x^{\alpha\beta}\Omega_{\beta\alpha} = 0$ and the frame $\theta_2 = 0$, $x_2 = \Gamma_0$, $y_2 = 1$ is preserved. The coefficient $3/2$ in δy is fixed from the requirement that the commutator of two supersymmetry transformations produce an SO(4) rotation of x and θ (this commutator vanishes on y itself, since we have broken down the R symmetry completely by setting $y_2 = 1$).

After this the case $d = 5$ becomes very similar to the case $d = 6$ $\mathcal{N} = (1,0)$. In particular, we find

$$\delta\theta = \begin{pmatrix} w-z & & & \\ & w-z' & & \\ & & w-z & \\ & & & w-z' \end{pmatrix} \epsilon + (\epsilon\theta\theta)\,; \tag{3.31}$$

$$\begin{aligned} \delta z &= \epsilon^1\theta^3 + \theta^1\epsilon^3\,, \\ \delta z' &= \epsilon^2\theta^4 + \theta^2\epsilon^4\,, \end{aligned} \tag{3.32}$$

so the superconformal completion of the variables z, z' is

$$\begin{aligned} \hat{z} &= z - \frac{\theta^1\theta^3}{w-z} + O((\theta)^4) \\ \hat{z}' &= z' - \frac{\theta^2\theta^4}{w-z'} + O((\theta)^4) \\ \hat{w} &= w - \frac{3}{2}\frac{\theta^1\theta^3}{w-z} - \frac{3}{2}\frac{\theta^2\theta^4}{w-z'} + O((\theta)^4)\,. \end{aligned} \tag{3.33}$$

As a consistency check we may consider the invariant (cf. (2.49) for $d = 4$)

$$\frac{y_{14}y_{23}}{y_{12}y_{34}} \left(\frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}\right)^{3/2} \quad \to \; w^{-1}(zz')^{3/2}\,. \tag{3.34}$$

It is easy to see that this combination remains regular if we use the variables (3.33), according to the general statement (3.27). Notice that the only difference between (3.28) for the case $d = 6$ $\mathcal{N} = (1,0)$ and (3.33) for the case $d = 5$ is in the coefficient in $\hat{w}$.

3.3.3 $d = 4$, $\mathcal{N} = 4$ and $\mathcal{N} = 2$

Here we are considering the superalgebras PSU(2,2|4) and SU(2,2|2), the supersymmetric extensions of the $d = 4$ conformal group SO(2,4), with R symmetry SU(4) $\sim$ SO(6) and SU(2) $\sim$ SO(3), respectively.

The case $d = 4$ $\mathcal{N} = 4$ is similar to the case $d = 6$ $\mathcal{N} = (2,0)$. We choose

$$\Gamma_0 = \sigma_0 = \delta^{\alpha\alpha'}\,, \quad \Gamma_1 = \sigma_3\,; \qquad \gamma_1 = \sigma_0 = \delta^{aa'}\,, \quad \gamma_2 = \sigma_3\,, \tag{3.35}$$

so primed and unprimed spinor indices (space-time and internal) become equivalent and their position (up or down) is irrelevant. The even coordinate matrices become diagonal[17]

$$x^{\alpha\beta} \equiv x_{12}^{\alpha\beta} = \begin{pmatrix} z & 0 \\ 0 & z' \end{pmatrix}, \qquad y^{ab} \equiv y_{12}^{ab} = \begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}. \tag{3.36}$$

[17] The fact that the variables z, z' and w, w' appear as the eigenvalues of the matrices $x^{\alpha\alpha'}$ and y^{ab} was first pointed out in [12].

The supersymmetry transformations involve two parameters, a left- and a right-handed one:

$$\begin{aligned}
\delta x^{\alpha\beta} &= \epsilon^{\alpha}{}_{b}\bar\theta^{b\beta} + \theta^{\alpha b}\bar\epsilon^{b}{}_{\beta} \\
\delta y^{ab} &= -\bar\epsilon^{a}{}_{\alpha}\theta^{\alpha b} + \bar\theta^{a\beta}\epsilon^{\beta}{}_{b} \\
\delta\theta^{\alpha b} &= \epsilon^{\alpha}{}_{a}y^{ab} - x^{\alpha\beta}\epsilon^{\beta}{}_{b}\,, \\
\delta\bar\theta^{a\beta} &= -\bar\epsilon^{a}{}_{\alpha}x^{\alpha\beta} + y^{ab}\bar\epsilon^{b}{}_{\beta}\,.
\end{aligned} \tag{3.37}$$

Once again, the form of these transformations follows from the preservation of the frame $\theta_2 = \bar\theta_2 = 0$, $x_2 = y_2 = \sigma_0$ and from the requirement that their commutator reduce to $\mathrm{SO}(3)\times\mathrm{SO}(3)$ transformations of the coordinates. In the fixed frame (3.36) we obtain the transformations

$$\begin{aligned}
\delta\begin{pmatrix}\theta^{11} & \theta^{12}\\ \theta^{21} & \theta^{22}\end{pmatrix} &= \begin{pmatrix}(w-z)\epsilon^{1}{}_{1} & (w'-z)\epsilon^{1}{}_{2}\\ (w-z')\epsilon^{2}{}_{1} & (w'-z')\epsilon^{2}{}_{2}\end{pmatrix} + (\epsilon\theta\theta) \\
\delta\begin{pmatrix}\bar\theta^{11} & \bar\theta^{12}\\ \bar\theta^{21} & \bar\theta^{22}\end{pmatrix} &= \begin{pmatrix}(w-z)\bar\epsilon^{1}{}_{1} & (w-z')\bar\epsilon^{1}{}_{2}\\ (w'-z)\bar\epsilon^{2}{}_{1} & (w'-z')\bar\epsilon^{2}{}_{2}\end{pmatrix} + (\epsilon\theta\theta)
\end{aligned} \tag{3.38}$$

along with

$$\begin{aligned}
\delta z &= \epsilon^{1}{}_{a}\bar\theta^{a1} + \theta^{1a}\bar\epsilon^{a}{}_{1} \\
\delta z' &= \epsilon^{2}{}_{a}\bar\theta^{a2} + \theta^{2a}\bar\epsilon^{a}{}_{2} \\
\delta w &= -\bar\epsilon^{1}{}_{\alpha}\theta^{\alpha 1} + \bar\theta^{1\alpha}\epsilon^{\beta}{}_{1} \\
\delta w' &= -\bar\epsilon^{2}{}_{\alpha}\theta^{\alpha 2} + \bar\theta^{2\alpha}\epsilon^{\beta}{}_{2}\,.
\end{aligned} \tag{3.39}$$

This yields the linearized superconformal completions

$$\begin{aligned}
\hat z &= z - \frac{\theta^{11}\bar\theta^{11}}{w-z} - \frac{\theta^{12}\bar\theta^{21}}{w'-z} + O((\theta)^4) \\
\hat z' &= z' - \frac{\theta^{21}\bar\theta^{12}}{w-z'} - \frac{\theta^{22}\bar\theta^{22}}{w'-z'} + O((\theta)^4) \\
\hat w &= w - \frac{\theta^{11}\bar\theta^{11}}{w-z} - \frac{\theta^{21}\bar\theta^{12}}{w-z'} + O((\theta)^4) \\
\hat w' &= w' - \frac{\theta^{12}\bar\theta^{21}}{w'-z} - \frac{\theta^{22}\bar\theta^{22}}{w'-z'} + O((\theta)^4)\,.
\end{aligned} \tag{3.40}$$

We note that the general relation (3.27) holds in the singular regime $w \to z$.

Finally, the reduction from $\mathcal{N}=4$ to $\mathcal{N}=2$ is achieved by suppressing the R symmetry indices a, b and by setting $w' = \infty$. In this way we find

$$\begin{aligned}
\hat z &= z - \frac{\theta^{1}\bar\theta^{1}}{w-z} + O((\theta)^4) \\
\hat z' &= z' - \frac{\theta^{2}\bar\theta^{2}}{w-z'} + O((\theta)^4) \\
\hat w &= w - \frac{\theta^{1}\bar\theta^{1}}{w-z} - \frac{\theta^{2}\bar\theta^{2}}{w-z'} + O((\theta)^4)\,.
\end{aligned} \tag{3.41}$$

3.3.4 $d = 3$, $\mathcal{N} = 8$

Here we are considering the superalgebra $\mathrm{OSp}(8|4, \mathbb{R})$, the supersymmetric extensions of the $d = 3$ conformal group $\mathrm{SO}(2, 3)$, with R symmetry $\mathrm{SO}(8)$. This case is the "mirror" of the case $d = 6$ $\mathcal{N} = (2, 0)$. Indeed, now the space-time variables form an $\mathrm{SO}(1, 2)$ vector, similar to the $\mathrm{SO}(3)$ internal vector in $d = 6$. The internal variables form an $\mathrm{SO}(6)$ vector which is the analog of the $\mathrm{SO}(1, 5)$ space-time vector in $d = 6$. The odd variables are the same, with the internal and space-time indices interchanged.

4. Superconformal Ward identities

The supersymmetrization of the four-point correlators (2.52), (2.53), (2.54) goes in two steps. Firstly, we replace $x_{ij}^2 \to \check{x}_{ij}^2$ in the propagator factors, as explained in section 3.2. This step does not create any harmonic singularities since the propagators are free from them. The non-trivial step is replacing the arguments z, z', w, w' of the invariant amplitude $G_k^{(\varepsilon)}$ by the (linearized) superconformal invariants found in section 3.3. The potential problem are the singularities present in (3.21), (3.28), (3.33), (3.40), (3.41). They occur when the internal variables w, w' approach the space-time variables z, z'. It is clear that such singularities must not take place. Indeed, the amplitude (2.53) is well defined for any value of w, w' (in fact, at any point in the space of the auxiliary internal variables y) and this must be so for its supersymmetric partners. Thus, we have to make sure that these singularities are suppressed.

In what follows we restrict ourselves to four-point functions of $\frac{1}{2}$-BPS operators of equal weights k. The generalization to four different weights is trivial. According to (2.42), it consists in multiplying the four-point amplitude with equal weights by a three-point function. The supersymmetrization of the latter is straightforward and does not create or remove any singularities.

4.1 Ward identities and their solution in the case $d = 4$ $\mathcal{N} = 2$

The superconformal completions of the invariant variables (3.41) become singular either when $w \to z$ or when $w \to z'$. Let us assume for the time being that $z \neq z'$. This allows us to treat the two singularities independently. The linearized $\mathcal{N} = 2$ transformations of the θs are obtained from (3.38):

$$\delta\theta^1 = (w - z)\,\epsilon^1 + (\epsilon\theta\theta)\,, \qquad \delta\theta^2 = \left(w - z'\right)\epsilon^2 + (\epsilon\theta\theta)\,. \tag{4.1}$$

We see that when, for instance, $w \to z' \neq z$ we are allowed to make an additional superconformal transformation which sets $\theta^1 = 0$.[18] In this frame $\Delta z = \hat{z} - z = 0$ and the only source of potential singularities are the nilpotent terms

$$\Delta z' = \Delta w = -\frac{\theta^2\bar{\theta}^2}{w - z'}\,, \qquad (\Delta z')^2 = (\Delta w)^2 = 0\,. \tag{4.2}$$

[18] Strictly speaking, we need to know whether the non-liner terms in (4.1) bring in new singularities of the same type. Adapting the $d = 6$ discussion around eq. (3.20), we can see that the singularity in the non-liner terms is of a different type, $(z - z')^{-1}$, which we do not consider for the moment.

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It is important to realize that in the new frame the linearized superconformal invariants (3.41) become exact, since there can be no non-linear terms in the single surviving nilpotent pair $\theta^2\bar\theta^2$. The possibility to fix such a frame while staying in the vicinity of one particular singular point, is crucial for our argument. This explains why we need not know the full non-linear version of (3.41) (see, however, appendix A for a discussion of the singularity $(z-z')^{-1}$).

Thus, the supersymmetrized invariant amplitude (2.51) gives rise to the following singular term:

$$G_k^{\mathcal{N}=2}(z,z'+\Delta z';w+\Delta w) = G_k^{\mathcal{N}=2}(z,z';w) - \frac{\theta^2\bar\theta^2}{w-z'}(\partial_{z'}+\partial_w)G_k^{\mathcal{N}=2}(z,z';w)\,. \tag{4.3}$$

It is then clear that the condition for suppressing the singularity is

$$(\partial_{z'}+\partial_w)G_k^{\mathcal{N}=2}(z,z';w)\big|_{w\to z'} = 0\,. \tag{4.4}$$

This constraint is what we call a *superconformal Ward identity* for $d=4$ $\mathcal{N}=2$.

Now, recall that $G_k^{\mathcal{N}=2}$ in (2.51) is a polynomial in the single variable zz'/w, made out of non-singular propagators, which has the property

$$(\partial_{z'}+\partial_w)\left(\frac{zz'}{w}\right) = 0\,. \tag{4.5}$$

So, condition (4.4) only affects the coefficient functions in (2.51):

$$\sum_{n=0}^{k}\partial_{z'}a_n(z,z')\left(\frac{zz'}{w}\right)^n\Bigg|_{w\to z'} = \partial_{z'}\sum_{n=0}^{k}a_n(z,z')z^n = 0\,. \tag{4.6}$$

In other words, the function

$$G_k^{\mathcal{N}=2}(z,z';z') = \sum_{n=0}^{k}a_n(z,z')z^n \equiv h(z) \tag{4.7}$$

must depend on the single variable z only. This condition is another form of the superconformal Ward identity (4.4).

Similarly, the examination of the singularity near $w\to z\neq z'$ leads to the superconformal Ward identity

$$(\partial_{z}+\partial_w)G_k^{\mathcal{N}=2}(z,z';w)\big|_{w\to z} = 0\,, \tag{4.8}$$

i.e., to the single variable function

$$G_k^{\mathcal{N}=2}(z,z';z) = \sum_{n=0}^{k}a_n(z,z')z'^n \equiv h(z')\,. \tag{4.9}$$

The fact that $h(z')$ is the same function as $h(z)$ follows from the exchange symmetry $a_n(z,z')=a_n(z',z)$ of the coefficients of the amplitude.

The contribution of the function $h(z)$ to the amplitude can be obtained by setting, e.g., $a_n=0$ for $n=2,\dots,k$. This is equivalent to considering the case $k=1$. The remaining two coefficients a_0, a_1 can be solved for from the linear equations (4.7), (4.9) in terms of $h(z)$ and $h(z')$, which gives

$$a_0 + a_1 \frac{zz'}{w} \;\to\; \frac{zh(z') - z'h(z)}{z - z'} + \frac{h(z) - h(z')}{z - z'}\,\frac{zz'}{w}\,. \tag{4.10}$$

We remark that although we have derived the Ward identity (4.4) under the assumption $z \neq z'$, the expression (4.10) remains well defined when $z \to z'$.

In (4.10) we only see the first two coefficients of the amplitude. Since the single variable contribution was obtained by studying the limit $w \to z'$ (or $w \to z$), it is clear that the remaining $k-1$ coefficients should form a contribution which vanishes in this limit. It can be put in the form of a *factorized polynomial*:

$$(w - z)(w - z') \sum_{n=0}^{k-2} A_n(z, z') \left(\frac{zz'}{w}\right)^{n+2} \tag{4.11}$$

(we recall that this is a polynomial in $1/w$ of degree k). It automatically solves the Ward identities (4.4) and (4.8). Indeed, expanding (4.11) in the powers of $1/w$ and expressing a_n, $n = 0, \ldots, k$ from (2.50) in terms of A_n, $n = 0, \ldots, k-2$, it is easy to see that the single variable function (4.7) identically vanishes. Another way to show that (4.11) is a solution to the Ward identities is to realize that the prefactor in (4.11) suppresses both singularities coming from the expansions of $\hat{z}$ and $\hat{z}'$ in $A_n(z, z')$. At the same time, the prefactor itself is regular, as follows from (4.2).

Combining eqs. (4.10) and (4.11), we can write down the complete solution to the superconformal Ward identities, i.e. the most general four-point amplitude of $\frac{1}{2}$-BPS $d = 4$ $\mathcal{N} = 2$ operators of weight k in the following form:

$$\begin{aligned} G_k^{\mathcal{N}=2}(z, z'; w) = &\ \frac{zh(z') - z'h(z)}{z - z'} + \frac{h(z) - h(z')}{z - z'}\,\frac{zz'}{w} + \\ &+ (w - z)(w - z') \sum_{n=0}^{k-2} A_n(z, z') \left(\frac{zz'}{w}\right)^{n+2}, \end{aligned} \tag{4.12}$$

where $h(z)$ is an arbitrary single variable function and $A_n(z, z') = A_n(z', z)$ is a set of $k-1$ arbitrary symmetric functions. Setting $w = z'$ or $w = z$ in (4.12) reproduces the definitions of the single variable functions (4.7), (4.9).

Finally, let us come back to the issue of singularities in the regime $z \to z'$. In this case both linearized supersymmetry transformations (4.1) may become singular simultaneously, therefore we are not allowed to shift away any of the θs. Moreover, the non-linear terms in $\delta\theta$ are singular when $z \to z'$ (recall (3.20) and the analogous terms for any d). In order to make sure that we have not missed any further constraints we ought to find the full non-linear version of (4.2) and then expand the amplitude (2.50) taking into account the Ward identity (4.4). A thorough analysis of this issue in the case $d = 4$ $\mathcal{N} = 4$ is presented in appendix A. Here we just give an example, restricting ourselves to the simplest case $d = 4$ $\mathcal{N} = 2$, weight $k = 2$, and considering only the factorized form (4.11) which represents the non-trivial (quantum) part of the amplitude:

$$\begin{aligned}(\hat{w}-\hat{z})(\hat{w}-\hat{z}')A(\hat{z},\hat{z}') &= (w-z)(w-z')A- \\ &-\theta^1\bar{\theta}^1(A+(w-z')A_z)-\theta^2\bar{\theta}^2(A+(w-z)A_{z'})+ \\ &+\theta^1\bar{\theta}^1\theta^2\bar{\theta}^2\left(A_{zz'}-\frac{A_{z'}-A_z}{z'-z}\right)\end{aligned} \tag{4.13}$$

(we have dropped the non-singular propagator factor). It is then clear that a sufficient condition which removes the singularity $(z'-z)^{-1}$ is that the coefficient function $A(z,z')$ admit an expansion in the even positive powers of $z'-z$. To put it differently, the function $A(z,z')$ should be well defined on the line $z=z'$ and in addition it should be symmetric in z,z', as postulated earlier. The examination of the singularity $(z'-z)^{-1}$ in the full amplitude for arbitrary k, including the single variable part $h(z)$, does not lead to any new conditions.

4.2 Ward identities and their solution in the case $d=4$ $\mathcal{N}=4$

As in the case $\mathcal{N}=2$, we first examine the singular behaviour under the assumption that $z\neq z'$. From the linearized transformations (3.38) of the θs it is clear that treating the harmonic singularities as isolated, for instance, taking $w'\to z'\neq z$ but keeping all other factors in (3.38) non-vanishing, a further frame fixing is possible in which the only non-vanishing odd variables are θ^{22} and $\bar{\theta}^{22}$. This reduces (3.40) to

$$\Delta z'=\Delta w'=-\frac{\theta^{22}\bar{\theta}^{22}}{w'-z'}\,,\qquad (\Delta z')^2=(\Delta w')^2=0\,. \tag{4.14}$$

Expanding the amplitude in $\Delta z',\Delta w'$ creates a simple pole at the coincident point $w'=z'$:

$$G_k^{\mathcal{N}=4}(z,z'+\Delta z';w,w'+\Delta w')=G_k^{\mathcal{N}=4}(z,z';w,w')-\frac{\theta^{22}\bar{\theta}^{22}}{w'-z'}(\partial_{z'}+\partial_{w'})G_k^{\mathcal{N}=4}(z,z';w,w')\,. \tag{4.15}$$

It is then clear that the condition for suppressing the singularity is

$$(\partial_{z'}+\partial_{w'})G_k^{\mathcal{N}=4}(z,z';w,w')\big|_{w'\to z'}=0\,. \tag{4.16}$$

The amplitude $G_k^{\mathcal{N}=4}$ in (2.45) is a polynomial in two non-singular propagator-type variables having the properties

$$(\partial_{z'}+\partial_{w'})\left(\frac{(1+w)(1+w')}{(1+z)(1+z')}\right)\bigg|_{w'\to z'}=0\,,\qquad (\partial_{z'}+\partial_{w'})\left(\frac{zz'}{ww'}\right)=0\,. \tag{4.17}$$

Thus, only the coefficient functions in (2.45) are affected by the constraint (4.16):

$$\sum_{0\le m+n\le k}\partial_{z'}a_{mn}(z,z')\left(\frac{(1+w)(1+w')}{(1+z)(1+z')}\right)^m\left(\frac{zz'}{ww'}\right)^{m+n}\Bigg|_{w'\to z'}=0\,. \tag{4.18}$$

In other words, the function

$$G_k^{\mathcal{N}=4}(z,z';w,z')=\sum_{0\le m+n\le k}a_{mn}(z,z')\frac{z^{m+n}}{(1+z)^m}\frac{(1+w)^m}{w^{m+n}}\equiv H(z,w) \tag{4.19}$$

must depend only on z (it is manifestly holomorphic in w as well). The dependence on w^{-1} in $H(z,w)$ is polynomial,

$$H(z,w)=\sum_{n=0}^{k} h_n(z)w^{-n}\,, \tag{4.20}$$

giving rise to $k+1$ single variable functions

$$h_n(z)=\sum_{p=0}^{n}\sum_{q=n}^{k} a_{q-p,p}(z,z')C_{q-p}^{n-p}\frac{z^q}{(1+z)^{q-p}}\,,\qquad n=0,\dots,k\,. \tag{4.21}$$

In fact, these functions are not all independent. Indeed, examining the singularity $w\to z$ and taking account of the exchange symmetry of the coefficient functions $a_{mn}(z,z')$ leads to the equivalent set of $k+1$ functions $H(z',w')=\sum_{n=0}^{k} h_n(z')w'^{-n}$. Then we find

$$H(z,z)=\sum_{n=0}^{k} h_n(z)z^{-n}=\sum_{0\le m+n\le k} a_{mn}(z,z')=H(z',z')=\sum_{n=0}^{k} h_n(z')z'^{-n}=\mathcal{C}\,, \tag{4.22}$$

where $\mathcal{C}$ is a constant.

The part of the correlator which is not affected by the Ward identity (4.21) (and its analogs obtained by exchange of variables) should vanish in all four singular limits. This suggests to cast it in the form of a *factorized polynomial*:

$$\begin{aligned}\mathcal{F}_k^{\mathcal{N}=4}(z,z';w,w') &= (w-z)(w'-z)(w-z')(w'-z')\times\\ &\times\sum_{0\le m+n\le k-2} A_{mn}(z,z')\left(\frac{(1+w)(1+w')}{(1+z)(1+z')}\right)^m\left(\frac{zz'}{ww'}\right)^{m+n+2}.\end{aligned} \tag{4.23}$$

Indeed, the prefactor in (4.23) suppresses all singularities coming from the expansions of $\hat{z}$ and $\hat{z}'$ in $A_{mn}(z,z')$. At the same time, the prefactor itself is regular, as follows from (3.40). In ref. [7] it has been shown that the quantum corrections to the correlator with $k=2$, generated through the field-theoretic insertion procedure, take the factorized form (4.23).[19]

So, we see that the $\frac{1}{2}(k+1)(k+2)$ coefficients $a_{mn}(z,z')$ of the amplitude (2.41) split into two subsets. One is given by the $\frac{1}{2}k(k-1)$ terms of the polynomial of degree $k-2$ in (4.23), and they are not restricted in any way by the superconformal Ward identities. The remaining $2k+1$ coefficients are expressed in terms of the $k+1$ single variable functions (4.21) subject to the condition (4.22), which makes k independent single variable functions. The complete amplitude can be written down as follows:

$$G_k^{\mathcal{N}=4}(z,z';w,w')=\mathcal{H}_k^{\mathcal{N}=4}(z,z';w,w')+\mathcal{F}_k^{\mathcal{N}=4}(z,z';w,w')\,, \tag{4.24}$$

where $\mathcal{H}_k^{\mathcal{N}=4}$ is the single variable contribution. As shown in [15], the latter can be cast in a form generalizing (4.10):

$$\begin{aligned}&\mathcal{H}_k^{\mathcal{N}=4}(z,z';w,w')=\\ &=-\mathcal{C}+\frac{(w-z)(w'-z')[H(z,w')+H(z',w)]-(w-z')(w'-z)[H(z,w)+H(z',w')]}{(z-z')(w-w')}\,,\end{aligned} \tag{4.25}$$

[19] For the generalization to higher weights see [13, 14]. The factorized form (4.23) in terms of variables of this type appeared in [12].

where the constant $\mathcal{C}$ has been defined in (4.22). Indeed, it clearly has the required exchange symmetry properties. Further, setting $w' = z'$ we immediately reproduce the definition (4.19) of $H(z,w)$ (or of $H(z',w')$ by setting $w = z$).

Finally, let us mention the issue of the singularity when $z \to z'$. In this regime it is not possible to treat the various singularities as independent. Then we need to expand the amplitude in the full non-linear $\Delta z, \Delta z'$ and to study the occurrences of poles in $z - z'$. To simplify the calculation, we have done this in the simplest case $k = 2$ and we have taken the amplitude in the factorized form (4.23). The details are given in appendix A. The conclusion is that the natural requirement of symmetry under the exchange of z and z' is sufficient to suppress all such singularities.

4.3 Ward identities in the cases $d = 3, 5, 6$

The treatment of the $d = 4$ singularities presented above can easily be generalized to all other dimensions. The case $d = 6$ $\mathcal{N} = (2,0)$ is similar to $d = 4$ $\mathcal{N} = 4$. Inspecting the linearized supersymmetry transformations (3.18), we see that choosing a particular singular limit, e.g., $w' \to z'$, affects only θ^{22} and θ^{42}. So, in this limit we can set to zero all θs but this pair. Consequently, the singular terms in (3.21) appear only in

$$\Delta w' = 2\Delta z' = -2\frac{\theta^{22}\theta^{42}}{w' - z'}\,, \qquad (\Delta z')^2 = (\Delta w')^2 = 0\,. \tag{4.26}$$

Thus, the linearized expressions (3.27) now become exact. Then the expansion of the general amplitude (2.53) terminates at level one and gives rise to a simple pole $(w' - z')^{-1}$. The condition for suppressing this pole is

$$(\partial_{z'} + \varepsilon\partial_{w'})G_k^{(\varepsilon)}(z,z';w,w')\Big|_{w'\to z'} = 0\,, \tag{4.27}$$

where $\varepsilon = d/2 - 1 = 2$. This is also the form of the constraint (4.16) for the case $d = 4$ $\mathcal{N} = 4$, with $\varepsilon = 1$. As mentioned earlier, the case $d = 3$ $\mathcal{N} = 8$ is the "mirror" of the case $d = 6$ $\mathcal{N} = (2,0)$ obtained by exchanging the space-time and internal sectors, which again leads to (4.27) with $\varepsilon = \frac{1}{2}$. Thus, eq. (4.27), together with its $z \leftrightarrow z'$, $w \leftrightarrow w'$ counterparts, are the superconformal Ward identities for any d and for R symmetry SO(n), $n > 3$.

In terms of the coefficient functions the constraint (4.27) becomes

$$\sum_{0\le m+n\le k} z'^{(\varepsilon-1)(m+n)}\,\partial_{z'}a_{mn}(z,z')\frac{z^{\varepsilon(m+n)}}{(1+z)^{\varepsilon m}}\,\frac{(1+w)^m}{w^{m+n}} = 0\,. \tag{4.28}$$

The crucial difference from the case $d = 4$ ($\varepsilon = 1$) is in the presence of the z'-dependent term. For $\varepsilon \neq 1$ it does not allow us to pull out the derivative, thus obtaining a set of single variable functions. The best we can do now is to expand in the powers of w^{-1} and derive a set of $k + 1$ differential constraints (Ward identities). The other singular limits imply analogous conditions which are obtained by exchanging the variables.

The cases $d = 6$ $\mathcal{N} = (1,0)$ and $d = 5$ are similar to $d = 4$ $\mathcal{N} = 2$. For instance, in $d = 5$, examining (3.31), (3.33) we see that $\Delta w = \frac{3}{2}\Delta z'$ in the singular limit $w \to z'$. This leads to the general superconformal Ward identities for the cases with R symmetry SO(3):

$$(\partial_{z'} + \varepsilon\partial_{w})G_k^{(\varepsilon)}(z,z';w)\Big|_{w\to z'} = 0\,, \tag{4.29}$$

and analogously with $z \leftrightarrow z'$. In terms of the coefficient functions (4.29) becomes

$$\sum_{n=0}^{k} z^{\varepsilon n} z'^{(\varepsilon-1)n} \partial_{z'} a_n(z,z') = 0\,. \tag{4.30}$$

For $d \neq 4$ solving the above constraints is a rather non-trivial task. In the special case $d = 6$ $\mathcal{N} = (2,0)$ with $k = 2$ this has been done in [16], making use of the crossing symmetry of the amplitude. In sections 5, 6 we give the general solution, for any k and any d, without crossing symmetry. As mentioned earlier, this is most conveniently done in terms of the variables (2.56), whereby the Ward identity (4.27) reads

$$(\chi\partial_\chi - \varepsilon\alpha\partial_\alpha) G_k^{(\varepsilon)}(\chi,\chi';\alpha,\alpha')\Big|_{\alpha\to 1/\chi} = 0\,, \tag{4.31}$$

and similarly for (4.29),

$$(\chi\partial_\chi - \varepsilon\alpha\partial_\alpha) G_k^{(\varepsilon)}(z,z';\alpha)\Big|_{\alpha\to 1/\chi} = 0\,. \tag{4.32}$$

5. Ward identity solutions for R symmetry SO(3)

In this section we consider the Ward identities for four-point functions involving $\frac{1}{2}$-BPS operators with R symmetry SO(3). This corresponds to the cases $d = 6$ $\mathcal{N} = (1,0)$, $d = 5$ and $d = 4$ $\mathcal{N} = 2$ discussed above. We want to solve the Ward identity (4.32) along with its partner equation under $\chi \leftrightarrow \chi'$, for general ε. In section 4.1 we did this only for $\varepsilon = 1$. We saw that the Ward identities effectively concern only the first two coefficients a_0, a_1 in the amplitude (see (4.10)), the rest remain unconstrained (see (4.11)). In fact, keeping only a_0, a_1 amounts to considering the case $k = 1$. This case is of no physical interest, since one cannot have a gauge invariant $\frac{1}{2}$-BPS operator of weight 1, which would be the elementary on-shell multiplet. Nevertheless, this case is very useful as a starting point in solving the Ward identities. It allowed us to introduce the single variable function h of section 4.1. Now we are going to the same, but this time for general ε. The next step will be to introduce the analog of (4.11), i.e. the arbitrary part of the amplitude which automatically solves the Ward identities. The crucial difference from the case $\varepsilon = 1$ will be that for $\varepsilon \neq 1$ we will be able to absorb the single variable functions into the unconstrained part of the amplitude.

In summary, the main result of this section is that

$$G_k^{(\varepsilon)}(\chi,\chi';\alpha)|_{\varepsilon\neq 1} = \sum_{n=0}^{k-2} (\chi\chi')^{(n+2)\varepsilon}\,\Delta_\varepsilon\,(\chi\alpha - 1)(\chi'\alpha - 1) b_n(\chi,\chi')\alpha^n\,, \tag{5.1}$$

in terms of a symmetric differential operator Δ_ε, defined later in section 5.2. Equation (5.1) may be rewritten in a basis independent way as in (5.39).

5.1 Solution for the case $k = 1$

The constraint equation (4.32) and its partner under $\chi \leftrightarrow \chi'$ become, for $k = 1$,

$$\frac{\partial}{\partial \chi} a_0 + \frac{u^\varepsilon}{\chi} \frac{\partial}{\partial \chi} a_1 = 0\,, \qquad \frac{\partial}{\partial \chi'} a_0 + \frac{u^\varepsilon}{\chi'} \frac{\partial}{\partial \chi'} a_1 = 0\,, \tag{5.2}$$

where $u = \chi\chi'$ (recall (2.3)). Clearly, the case $\varepsilon = 1$ is exceptional, since, e.g., $u/\chi = \chi'$ and we can pull out the derivative ∂_χ. This immediately leads to the solution in the form of a single variable function [11]. However, the situation is not as simple as this for $\varepsilon \neq 1$. Now we are going to follow the method first used in [7] for $\varepsilon = 1$ and then in [16] for $\varepsilon = 2$. It consists of regarding (5.2) as a pair of coupled partial differential equations, whose integrability conditions can be written down in the form

$$D_\varepsilon \, \tilde{a}_0 = 0\,, \qquad D_\varepsilon \, a_1 = 0\,, \tag{5.3}$$

where $a_0 = u^\varepsilon \, \tilde{a}_0$ and the symmetric differential operator is given by

$$D_\varepsilon = \frac{\partial^2}{\partial \chi \, \partial \chi'} - \varepsilon \, \frac{1}{\chi - \chi'} \left(\frac{\partial}{\partial \chi} - \frac{\partial}{\partial \chi'} \right). \tag{5.4}$$

It satisfies the obvious identities

$$\frac{1}{\chi} \, \partial_{\chi'} \, u^\varepsilon \partial_\chi - \chi \leftrightarrow \chi' = -(\chi - \chi') u^{\varepsilon - 1} D_\varepsilon\,, \qquad \chi \, \partial_{\chi'} \, u^{-\varepsilon} \partial_\chi u^\varepsilon - \chi \leftrightarrow \chi' = -(\chi - \chi') D_\varepsilon\,, \tag{5.5}$$

with the help of which the consistency conditions for (5.2) can be written in the form (5.3).

To solve either of (5.3) we make the change of variables,

$$p = \chi + \chi'\,, \qquad q = \chi - \chi'\,, \tag{5.6}$$

whereby we may easily find

$$D_\varepsilon = \frac{\partial^2}{\partial p^2} - \frac{\partial^2}{\partial q^2} - 2\varepsilon \frac{1}{q} \frac{\partial}{\partial q}\,. \tag{5.7}$$

This form of the operator allows for solutions by separation of variables to (5.3). Writing either of $\tilde{a}_0$ or a_1 as $f(p)g(q)$ then (5.3) implies that

$$\frac{f''(p)}{f(p)} = \frac{1}{g(q)} \Big(g''(q) + 2\varepsilon \, \frac{1}{q} \, g'(q) \Big) = -c^2\,, \tag{5.8}$$

where c^2 is a real constant and where, due to the symmetry condition $a_j(\chi, \chi') = a_j(\chi', \chi)$, the function $g(q)$ must be even, $g(q) = g(-q)$. From here we may easily find for $f(p)$ that

$$f(p) = A(c) \cos cp + B(c) \sin cp\,, \tag{5.9}$$

while for $g(q)$ we have that

$$g(q) = C(c) q^{\frac{1}{2} - \varepsilon} \, J_{\frac{1}{2} - \varepsilon}(cq) + D(c) q^{\frac{1}{2} - \varepsilon} \, Y_{\frac{1}{2} - \varepsilon}(cq)\,, \tag{5.10}$$

involving Bessel functions of the first and second kind, respectively. Taking into account the evenness condition $g(q) = g(-q)$ and the properties of the Bessel functions, we conclude that $C(c) = 0$ for $\varepsilon = 1, 2, \ldots$ (corresponding to even dimensions) and $D(c) = 0$ for $\varepsilon = \frac{1}{2}, \frac{3}{2}, \ldots$ (corresponding to odd dimensions). For $\tilde{a}_0 = h_0^{(\varepsilon)}$, $a_1 = h_1^{(\varepsilon)}$, we may write the full solution to (5.3) as a superposition of (5.9) and (5.10),

$$h_j^{(\varepsilon)}(\chi, \chi') = (\chi - \chi')^{\frac{1}{2}-\varepsilon} \int \mathrm{d}c \big(A_j^{(\varepsilon)}(c) \cos(c\chi + c\chi') + B_j^{(\varepsilon)}(c) \sin(c\chi + c\chi') \big) \times \\ \times \begin{cases} Y_{\frac{1}{2}-\varepsilon}(c\chi - c\chi') & \text{for } \varepsilon = 1, 2, \ldots \\ J_{\frac{1}{2}-\varepsilon}(c\chi - c\chi') & \text{for } \varepsilon = \frac{1}{2}, \frac{3}{2}, \ldots \end{cases},$$

with arbitrary coefficients $A_j^{(\varepsilon)}(c)$, $B_j^{(\varepsilon)}(c)$.

This solution can be expressed in terms of two basic solutions, the one for $\varepsilon = 1$ and the other for $\varepsilon = \frac{1}{2}$:

$$\begin{aligned} h_j^{(\varepsilon)}(\chi, \chi') &= (D_\varepsilon)^{\varepsilon-1} h_j^{(1)}(\chi, \chi'), \qquad \varepsilon = 1, 2, \ldots, \\ h_j^{(\varepsilon)}(\chi, \chi') &= (D_\varepsilon)^{\varepsilon-\frac{1}{2}} h_j^{(\frac{1}{2})}(\chi, \chi'), \qquad \varepsilon = \frac{1}{2}, \frac{3}{2}, \ldots. \end{aligned} \tag{5.11}$$

To prove this we may use the recurrence relations

$$J_{n+1}(q) + J_{n-1}(q) = \frac{2n}{q} J_n(q), \qquad Y_{n+1}(q) + Y_{n-1}(q) = \frac{2n}{q} Y_n(q),$$

to show that

$$D_\varepsilon \mathrm{K}_n(p, q) = -\big(2(\varepsilon + n) - 1\big) c^2\, \mathrm{K}_{n-1}(p, q),$$

where $\mathrm{K}_n(p, q) = q^n (A(c) \cos cp + B(c) \sin cp) \big(J_n(cq), Y_n(cq)\big)$. It is easy to derive from this that

$$(A_j^{(\varepsilon)}(c),\, B_j^{(\varepsilon)}(c)) = (\varepsilon - 1)!(-2c^2)^{\varepsilon-1} (A_j^{(1)}(c),\, B_j^{(1)}(c))$$

for integer ε, and

$$(A_j^{(\varepsilon)}(c),\, B_j^{(\varepsilon)}(c)) = (\varepsilon - \frac{1}{2})!(-2c^2)^{\varepsilon-\frac{1}{2}} (A_j^{(\frac{1}{2})}(c),\, B_j^{(\frac{1}{2})}(c))$$

for half-integer ε.

Let us come back for a moment to the particular case $\varepsilon = 1$. We already know the solution in this case (recall (4.10)). Here we can reproduce it by substituting $h_j^{(1)}(\chi, \chi') \to \varphi(\chi, \chi')/(\chi - \chi')$ in (5.3), which immediately leads to

$$h_j^{(1)}(\chi, \chi') = \frac{h_j(\chi) - h_j(\chi')}{\chi - \chi'}, \tag{5.14}$$

for arbitrary h_j. It is not hard to see that the general solution (5.11) reduces to just (5.14) if $\varepsilon = 1$. Indeed, the Bessel function $Y_{-1/2}(t) \sim t^{-1/2} \sin t$, so (5.11) becomes the Fourier integral representation of (5.14).

In section 6.1 we find a similar result as in (5.11) for the $k=1$ case with SO(n) R symmetry, but this time in the form of expansions in terms of Jack polynomials whose properties are reviewed in appendix B. There also an alternative and convenient expansion is given for $h_j^{(\frac{1}{2})}(\chi,\chi')$ in terms of Legendre polynomials.

Focusing now on the integer ε case[20], further conditions on $h_0(\chi)$, $h_1(\chi)$ of (5.14) are implied by (5.2). For $\varepsilon=1$ it is easy to see that (cf. (4.10))

$$h_0(\chi) = C - \frac{1}{\chi}h_1(\chi) \quad \Rightarrow \quad G_1^{(1)}(\chi,\chi';\alpha) = u\,\frac{(\chi\alpha-1)h(\chi)-(\chi'\alpha-1)h(\chi')}{\chi-\chi'}\,, \tag{5.15}$$

where C is a constant and $h(\chi) = h_1(\chi)/\chi$. We may find similar conditions for other integer ε so that,

$$G_1^{(\varepsilon)}(\chi,\chi';\alpha)\big|_{\varepsilon=1,2,\ldots} = u^\varepsilon (D_\varepsilon)^{\varepsilon-1}\,\frac{(\chi\alpha-1)h(\chi)-(\chi'\alpha-1)h(\chi')}{\chi-\chi'}\,, \tag{5.16}$$

for arbitrary $h(\chi)$.

Using Jack polynomials we may unify even and odd dimensional results for the $k=1$ case, as shown in section 6. Here we quote the result which is

$$G_1^{(\varepsilon)}(\chi,\chi';\alpha) = u^\varepsilon \mathcal{H}^{(\varepsilon)}(\chi,\chi';\alpha)\,, \tag{5.17}$$

where

$$\mathcal{H}^{(\varepsilon)}(\chi,\chi';\alpha) = \sum_\lambda a_\lambda \left(\tilde{P}_{\lambda-1}^{(\varepsilon)}(\chi,\chi') - \alpha\,\tilde{P}_{\lambda}^{(\varepsilon)}(\chi,\chi')\right), \tag{5.18}$$

with arbitrary a_λ and where we define

$$\tilde{P}_\lambda^{(\varepsilon)}(\chi,\chi') = \frac{(2\varepsilon)_\lambda}{\Gamma(\lambda+1)}\,P_{\lambda\,0}^{(\varepsilon)}(\chi,\chi')\,, \tag{5.19}$$

where the Jack polynomials $P_{\lambda_1\lambda_2}^{(\varepsilon)}(\chi,\chi')$ are defined in appendix B. Here also we are using the Pochhammer symbol, $(a)_b = \Gamma(a+b)/\Gamma(b)$. We may reproduce (5.16) via (5.18) through,

$$\tilde{P}_\lambda^{(\varepsilon)}(\chi,\chi') = (-1)^{\varepsilon+1}\frac{1}{(\varepsilon-1)!^2}\,(D_\varepsilon)^{\varepsilon-1}\Big(\frac{\chi^{\lambda+2\varepsilon-1}-{\chi'}^{\lambda+2\varepsilon-1}}{\chi-\chi'}\Big)\,, \tag{5.20}$$

which follows from results in appendix B.

5.2 Solution for the general k case.

The first non-trivial case of a correlator corresponds to $k=2$, i.e., to bilinear gauge invariant composite $\frac{1}{2}$-BPS operators. The Ward identities we need to solve are given by

$$\partial_\chi a_0 + \frac{u^\varepsilon}{\chi}\partial_\chi a_1 + \frac{u^{2\varepsilon}}{\chi^2}\partial_\chi a_2 = 0\,, \qquad \partial_{\chi'} a_0 + \frac{u^\varepsilon}{\chi'}\partial_{\chi'} a_1 + \frac{u^{2\varepsilon}}{\chi'^2}\partial_{\chi'} a_2 = 0\,. \tag{5.21}$$

[20] The technically more involved half-integer case is dealt with in section 6 where the analysis simplifies in terms of Jack polynomials.

Let us recall once more the situation in four dimensions. There (5.21) becomes

$$\partial_\chi a_0 + \chi' \partial_\chi a_1 + \chi'^2 \partial_\chi a_2 = 0\,, \qquad \partial_{\chi'} a_0 + \chi \partial_{\chi'} a_1 + \chi^2 \partial_{\chi'} a_2 = 0\,. \tag{5.22}$$

We have two partial differential equations for three unknown functions, so one of the functions must remain arbitrary. Indeed, after the substitutions [4]

$$\tilde{a}_0 = a_0 - u\, a_2\,, \qquad \tilde{a}_1 = a_1 + (\chi + \chi')a_2 \tag{5.23}$$

the equations in (5.22) are reduced to those from the $k = 1$ case (5.2). The possibility to eliminate the coefficient a_2 from the constraints follows from the form (4.11) of the unconstrained part of the amplitude, which, for $k = 2$, is simply $(\chi\alpha-1)(\chi'\alpha-1)u^2 b(\chi,\chi')$. Given an amplitude which satisfies the Ward identities, we can always obtain another one by adding such a polynomial. We can use this freedom by choosing

$$a_2(\chi,\chi') = u\, b(\chi,\chi')\,, \tag{5.24}$$

and subtracting $(\chi\alpha - 1)(\chi'\alpha - 1)\, u a_2$ from the original $k = 2$ amplitude. Thus, we obtain a new amplitude with $\tilde{a}_2 = 0$ (i.e., of the $k = 1$ type) and with $\tilde{a}_0$, $\tilde{a}_1$ as given in (5.23).

We can apply the same strategy to (5.21) with general ε. The analog of (5.24) now is

$$a_2(\chi,\chi') = \Delta_\varepsilon\, u\, b(\chi,\chi')\,, \tag{5.25}$$

where $b(\chi,\chi') = b(\chi',\chi)$ is arbitrary. The operator

$$\Delta_f = (D_\varepsilon)^{f-1}\, u^{f-1}\,, \tag{5.26}$$

has Jack polynomials as eigenfunctions and is relevant to the subsequent expansions in terms of such polynomials.

It is important to realize that the substitution (5.25) is in general not invertible, except for $\varepsilon = 1$, where it takes the form (5.24). For $\varepsilon \neq 1$, the operator Δ_ε has a non-trivial kernel, which means that the function $b(\chi,\chi')$ is determined up to some freedom. We shall come back to this point shortly.

We may then rewrite (5.21) as

$$\partial_\chi (a_0 - u^{2\varepsilon} \Delta_\varepsilon b) + \frac{u^\varepsilon}{\chi}\, \partial_\chi (a_1 + u^\varepsilon\, \Delta_\varepsilon\, (\chi + \chi')b) = 0\,, \tag{5.27}$$

which reduces to the $k = 1$ case (5.2) again. In deriving (5.27) we have used the operator identity

$$\frac{u^{j\varepsilon}}{\chi^j}\, \partial_\chi\, (D_\varepsilon)^{\varepsilon-1}\, u = \frac{u^{(j-1)\varepsilon}}{\chi^{j-1}}\, \partial_\chi\, u^\varepsilon\, (D_\varepsilon)^{\varepsilon-1} (\chi + \chi') - \frac{u^{(j-2)\varepsilon}}{\chi^{j-2}}\, \partial_\chi\, u^{2\varepsilon}\, (D_\varepsilon)^{\varepsilon-1}\,, \tag{5.28}$$

which may be extended to cases of half-integer ε through use of Jack polynomial expansion (see section 6).

The solution to (5.27) is already known from section 5.1. We may write it as

$$a_0 = u^{2\varepsilon}\, \Delta_\varepsilon\, b + u^\varepsilon\, h_0^{(\varepsilon)}\,, \quad a_1 = -u^\varepsilon\, \Delta_\varepsilon\, (\chi + \chi')b + h_1^{(\varepsilon)}\,, \tag{5.29}$$

where $h_j^{(\varepsilon)}$ are given by (5.11).

With (5.25) and (5.29) we may write the complete solution to the $k=2$ Ward identities as

$$G_2^{(\varepsilon)}(\chi,\chi';\alpha) = u^{2\varepsilon}\Delta_\varepsilon(\chi\alpha-1)(\chi'\alpha-1)b(\chi,\chi') + u^\varepsilon \mathcal{H}^{(\varepsilon)}(\chi,\chi';\alpha)\,, \tag{5.30}$$

with $\mathcal{H}^{(\varepsilon)}(\chi,\chi';\alpha)$ given by (5.18). This form of the solution resembles the four-dimensional case (4.12) with $k=2$. Indeed, there we also have two terms, one involving a restricted (single variable) function (related to $\mathcal{H}^{(1)}$ in (5.30)) and another involving an arbitrary function of two variables ($b(\chi,\chi')$ in (5.30)). However, there is a crucial difference between the case $\varepsilon=1$ and the general case $\varepsilon\neq 1$. It has to do with the freedom in $b(\chi,\chi')$ mentioned above. It allows us to absorb the term $\mathcal{H}^{(\varepsilon)}$ into a redefinition of $b(\chi,\chi')$. Let us first illustrate with the case $\varepsilon=2$.

For $\varepsilon=2$, upon making the following redefinition in (5.25),

$$b\to b-b'\,,\qquad b'(\chi,\chi') = -\frac{1}{u^2}D_2\left(\frac{g(\chi)-g(\chi')}{\chi-\chi'}\right), \tag{5.31}$$

where $\Delta_\varepsilon\, ub'=0$, so that the modification b' has no effect on a_2, then we may find that,

$$u^4\Delta_2(\chi\alpha-1)(\chi'\alpha-1)b' = u^2 D_2\left(\frac{(\chi\alpha-1)h(\chi)-(\chi'\alpha-1)h(\chi')}{\chi-\chi'}\right), \tag{5.32}$$

where

$$-\chi g'(\chi)+2g(\chi) = h(\chi)\,. \tag{5.33}$$

determing g in terms of h. Of course, the right hand side of (5.32) is precisely of the form which cancels $\mathcal{H}^{(2)}$ in (5.29) as given in (5.17) with (5.16). This may be generalized most easily to other ε in terms of Jack polynomial expansions - we show this in section 6.

Now we are ready to move to the general k case. The idea is the same: we redefine the last $k-1$ coefficients of the amplitude $a_2,\dots,a_k$ in such a way that they all drop out from the constraints (4.32), thus again reducing the problem to the $k=1$ case. Using (5.28) and setting

$$\begin{aligned}
a_k &= \Delta_\varepsilon\, u\, b_{k-2}\,,\\
a_{k-1} &= \Delta_\varepsilon\, u\, b_{k-3} - u^\varepsilon\,\Delta_\varepsilon\,(\chi+\chi')\, b_{k-2}\,,\\
a_{k-2} &= \Delta_\varepsilon\, u\, b_{k-4} - u^\varepsilon\,\Delta_\varepsilon\,(\chi+\chi')\, b_{k-3} + u^{2\varepsilon}\,\Delta_\varepsilon\, b_{k-2}\,,\\
&\ \vdots\\
a_2 &= \Delta_\varepsilon\, u\, b_0 - u^\varepsilon\,\Delta_\varepsilon\,(\chi+\chi')\, b_1 + u^{2\varepsilon}\,\Delta_\varepsilon\, b_2\,,
\end{aligned} \tag{5.34}$$

where $b_i(\chi,\chi')=b_i(\chi',\chi)$ are arbitrary, we may reduce (4.32) to the $k=1$ case with

$$\partial_\chi\big(a_0-u^{2\varepsilon}\Delta_\varepsilon\, b_0\big) + \frac{1}{\chi}u^\varepsilon\,\partial_\chi\big(a_1+u^\varepsilon\,\Delta_\varepsilon(\chi+\chi')b_0 - u^{2\varepsilon}\Delta_\varepsilon b_1\big) = 0\,, \tag{5.35}$$

so that

$$\begin{aligned}
a_1 &= -u^\varepsilon\,\Delta_\varepsilon\,(\chi+\chi')b_0 + u^{2\varepsilon}\Delta_\varepsilon\, b_1 + h_1^{(\varepsilon)}(\chi,\chi')\,,\\
a_0 &= u^{2\varepsilon}\,\Delta_\varepsilon\, b_0 + h_0^{(\varepsilon)}(\chi,\chi')\,.
\end{aligned} \tag{5.36}$$

With (5.34) and (5.36) we have that,

$$G_k^{(\varepsilon)}(\chi,\chi';\alpha)=\sum_{i=0}^{k-2}u^{(i+2)\varepsilon}\alpha^i\,\Delta_\varepsilon\,(\chi\alpha-1)(\chi'\alpha-1)b_i(\chi,\chi')+u^\varepsilon\mathcal{H}^{(\varepsilon)}(\chi,\chi';\alpha)\,. \tag{5.37}$$

For $\varepsilon=1$ and with (5.16) this reproduces the general solution (4.12). We stress once more the fact that in four dimensions the single variable part of the solution (4.12) cannot be absorbed into the unconstrained part, unlike all other dimensions. From here (5.1) follows.

We may in fact write (5.1) in a basis independent way by defining the operator,

$$\mathcal{D}_\varepsilon=\left(D_\varepsilon+\varepsilon^2\frac{1}{u}\frac{\partial^2}{\partial\alpha^2}\,\alpha^2-\varepsilon\frac{1}{u\alpha}\,D_1^+\,\frac{\partial}{\partial\alpha}\,\alpha^2\right)^{\varepsilon-1}\,, \tag{5.38}$$

where D_1^+ is the Euler operator defined in appendix B, whereby we may find that,

$$G_k^{(\varepsilon)}(\chi,\chi';\alpha)|_{\varepsilon\neq1}=\left(\alpha^2\mathcal{D}_\varepsilon\,u-\alpha\mathcal{D}_\varepsilon\,(\chi+\chi')+\mathcal{D}_\varepsilon\right)\mathcal{F}_{k-2}^{(\varepsilon)}(\chi,\chi';\alpha)|_{\varepsilon\neq1}\,, \tag{5.39}$$

for arbitrary $\mathcal{F}_{k-2}^{(\varepsilon)}(\chi,\chi';\alpha)$ of degree $k-2$ in α, where we have used the fact that $\mathcal{H}^{(\varepsilon)}(\chi,\chi';\alpha)$ may be absorbed by b_0 in (5.37) for $\varepsilon\neq1$. (5.38) reduces to the unrestricted two-variable part of the four-dimensional solution when $\mathcal{D}_\varepsilon\to1$.

6. Ward identity solutions for R symmetry SO(n)

For general k, the main result of this section is that

$$\begin{aligned}G_k^{(\varepsilon)}(\chi,\chi';\alpha,\alpha')|_{\varepsilon\neq1}=&\sum_{0\leq n+m\leq k-2}u^{(m+n+2)\varepsilon}\,v^{-m\varepsilon}\,(\alpha\alpha')^n\,\big((\alpha-1)(\alpha'-1)\big)^m\times\\&\times\Delta_\varepsilon(\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)\mathcal{F}_{nm}(\chi,\chi')+\\&+u^\varepsilon\,\mathcal{H}_1^{(\varepsilon)}(\chi,\chi';\alpha,\alpha')\,,\end{aligned} \tag{6.1}$$

where $\mathcal{F}_{nm}$ are unrestricted two-variable functions and $\mathcal{H}_1^{(\varepsilon)}$ is a restricted solution to the Ward identities (which in even dimensions may be expressed in terms of single variable functions). The unrestricted part of (6.1) may be rewritten in a basis independent way as in (6.93).

6.1 Solutions for Ward identities in terms of Jack polynomials

Jack polynomials [21] provide a useful basis for the symmetric functions of χ, χ' of the four-point function with which to find solutions to the Ward identities, particularly in odd dimensions. Expansions in terms of Schur polynomials (the four dimensional reduction of Jack polynomials) have been used before in [12] enabling $d=4$, $\mathcal{N}=4$ superconformal four-point functions to be found. In this section we use Jack polynomials to find the $k=1$ solutions for SO(n) R symmetry. This demonstrates their convenience in unifying solutions in different dimensions. An important consequence of the analysis is that such solutions involve expansion in terms of a single set of Jack polynomials, $\tilde{P}_\lambda^{(\varepsilon)}(\chi,\chi')$ in (5.19). This is a feature of the general k case also whereby, upon using a technique similar to the SO(3) case to reduce the equations to a simpler set, they solve reduced sets of equations.

Also in this section we demonstrate the claim made about absorption of $\mathcal{H}^{(\varepsilon)}$ in (5.37) and prove (5.28) (which is used further in section 6.2 and appendix C), for general ε, using such expansions. Both rely on an important property of the operator Δ_ε in (5.26) - it has Jack polynomials as eigenfunctions. The proof of (5.28) also relies upon the solution to an important recurrence relation, (6.17), in section 6.1.1.

6.1.1 The $k = 1$ solutions in terms of Jack polynomials

For $k = 1$, we may find four independent equations coming from the Ward identities (4.31) constraining a_{00}, a_{10}, a_{01} given by,

$$\begin{aligned} \partial_\chi a_{00} + \frac{\chi-1}{\chi}\Big(\frac{u}{v}\Big)^\varepsilon \partial_\chi a_{01} &= 0\,, \qquad & \partial_{\chi'} a_{00} + \frac{\chi'-1}{\chi'}\Big(\frac{u}{v}\Big)^\varepsilon \partial_{\chi'} a_{01} &= 0\,, \\ \partial_\chi a_{01} + \frac{1}{1-\chi}\, v^\varepsilon\, \partial_\chi a_{10} &= 0\,, & \partial_{\chi'} a_{01} + \frac{1}{1-\chi'}\, v^\varepsilon\, \partial_{\chi'} a_{10} &= 0\,, \end{aligned}$$

which we may re-write more conveniently by defining,

$$a_{00} = u^\varepsilon\, \hat{a}_{00}\,, \qquad a_{01} = v^\varepsilon\, \hat{a}_{01}\,, \tag{6.3}$$

so that from the first set of equations in (6.2) we have that

$$\begin{aligned} \big(\varepsilon + \chi\partial_\chi\big)\hat{a}_{00} + \big(\varepsilon + (\chi-1)\partial_\chi\big)\hat{a}_{01} &= 0\,, \\ \big(\varepsilon + \chi'\partial_{\chi'}\big)\hat{a}_{00} + \big(\varepsilon + (\chi'-1)\partial_{\chi'}\big)\hat{a}_{01} &= 0\,, \end{aligned} \tag{6.4}$$

while from the second set we have that

$$\begin{aligned} \partial_\chi a_{10} - \big(\varepsilon + (\chi-1)\partial_\chi\big)\hat{a}_{01} &= 0\,, \\ \partial_{\chi'} a_{10} - \big(\varepsilon + (\chi'-1)\partial_{\chi'}\big)\hat{a}_{01} &= 0\,. \end{aligned} \tag{6.5}$$

Considering integrability constraints on (6.2) we may find that these imply that

$$D_\varepsilon\, \hat{a}_{00} = 0\,, \qquad D_\varepsilon\, \hat{a}_{01} = 0\,, \qquad D_\varepsilon\, a_{10} = 0\,, \tag{6.6}$$

for which we know the solutions from section 5.1. Further constraints on these solutions are implied by (6.4) and (6.5) however it proves easiest to satisfy these constraints through expansions as below. We may solve for these constraints quite simply for $\varepsilon = 2$ (corresponding to six dimensions) as the solution to (6.6) in this case is given by,

$$\hat{a}_{00} = D_2\bigg(\frac{g_1(\chi)-g_1(\chi')}{\chi-\chi'}\bigg)\,, \quad \hat{a}_{01} = D_2\bigg(\frac{g_2(\chi)-g_2(\chi')}{\chi-\chi'}\bigg)\,, \quad a_{10} = D_2\bigg(\frac{g_3(\chi)-g_3(\chi')}{\chi-\chi'}\bigg)\,, \tag{6.7}$$

for arbitrary $g_i(\chi)$ and to satisfy (6.4) we find that,

$$g_1(\chi) = -\frac{\chi-1}{\chi}g_2(\chi) + A_0\frac{1}{\chi} + A_1 + A_2\,\chi + A_3\,\chi^2\,, \tag{6.8}$$

for A_i being constants, while to satisfy (6.5) we find that,

$$g_3(\chi) = (\chi-1)g_2(\chi) + B_1 + B_2\,\chi + B_3\,\chi^2 + B_4\,\chi^3\,, \tag{6.9}$$

for B_i being constants.[21] Inserting these expression back into (6.7) we find that the A_i, B_i, $i=1,2,3$, contributions vanish irrespective of the choice of these constants. With (6.7), (6.8) and (6.9) we may find that

$$G_1^{(2)}(\chi,\chi';\alpha,\alpha') = u^2\, D_2\bigg(\frac{(\chi\alpha-1)(\chi\alpha'-1)h(\chi)-(\chi'\alpha-1)(\chi'\alpha'-1)h(\chi')}{\chi-\chi'}\bigg)\,, \tag{6.10}$$

where[22]

$$h(\chi) = \frac{1}{\chi}g_2(\chi) + A_0\frac{1}{\chi} + B_4\chi\,. \tag{6.11}$$

For general ε we now expand,

$$\hat{a}_{00} = \sum_{\lambda_1,\lambda_2} a_{\lambda_1\lambda_2}\,P^{(\varepsilon)}_{\lambda_1\lambda_2}\,, \qquad \hat{a}_{01} = \sum_{\lambda_1,\lambda_2} b_{\lambda_1\lambda_2}\,P^{(\varepsilon)}_{\lambda_1\lambda_2}\,, \qquad a_{10} = \sum_{\lambda_1,\lambda_2} c_{\lambda_1\lambda_2}\,P^{(\varepsilon)}_{\lambda_1\lambda_2}\,, \tag{6.12}$$

where $a_{\lambda_1\lambda_2}$, $b_{\lambda_1\lambda_2}$, $c_{\lambda_1\lambda_2}$, are arbitrary, so that we may easily obtain from (6.4) (using relations from appendix B),

$$\begin{aligned}(\lambda_1+\lambda_2+\varepsilon)(a_{\lambda_1\lambda_2}+b_{\lambda_1\lambda_2})&-\\ -\frac{(\lambda_1+\varepsilon+1)(\lambda_-+1)}{\lambda_-+\varepsilon+1}\,b_{\lambda_1+1\,\lambda_2}-\frac{(\lambda_2+1)(\lambda_-+2\varepsilon-1)}{\lambda_-+\varepsilon-1}\,b_{\lambda_1\,\lambda_2+1}&=0\,,\end{aligned} \tag{6.13}$$

and,[23]

$$\begin{aligned}(\lambda_1+\lambda_2)b_{\lambda_1\lambda_2}-\frac{(\lambda_1+\varepsilon-1)(\lambda_-+2\varepsilon-1)}{\lambda_-+\varepsilon-1}(a_{\lambda_1-1\,\lambda_2}+b_{\lambda_1-1\,\lambda_2})&-\\ -\frac{(\lambda_2-1)(\lambda_-+1)}{\lambda_-+\varepsilon+1}(a_{\lambda_1\,\lambda_2-1}+b_{\lambda_1\,\lambda_2-1})&=0\,.\end{aligned} \tag{6.14}$$

Similarly, from (6.5) we may obtain,

$$\begin{aligned}(\lambda_1+\lambda_2+2\varepsilon)a_{\lambda_1\lambda_2}&+\\ +\frac{(\lambda_1+\varepsilon+1)(\lambda_-+1)}{\lambda_-+\varepsilon+1}c_{\lambda_1+1\,\lambda_2}+\frac{(\lambda_2+1)(\lambda_-+2\varepsilon-1)}{\lambda_-+\varepsilon-1}c_{\lambda_1\,\lambda_2+1}&=0\,,\end{aligned} \tag{6.15}$$

and

$$(\lambda_1+\lambda_2)c_{\lambda_1\lambda_2}+\frac{(\lambda_1+\varepsilon-1)(\lambda_-+2\varepsilon-1)}{\lambda_-+\varepsilon-1}a_{\lambda_1-1\,\lambda_2}+\frac{(\lambda_2-1)(\lambda_-+1)}{\lambda_-+\varepsilon+1}a_{\lambda_1\,\lambda_2-1}=0\,. \tag{6.16}$$

[21] One way of seeing (6.8), (6.9) is to expand (6.4), (6.5) in terms of either χ or χ' and $\chi-\chi'$ whereby, on Taylor expanding in $\chi-\chi'$, the coefficients of $(\chi-\chi')^{n-4}$, $n\geq 4$, are proportional, respectively, to

$$\begin{aligned} n\big(g_1^{(n-1)}(\chi)+g_2^{(n-1)}(\chi)\big)+\chi g_1^{(n)}(\chi)+(\chi-1)g_2^{(n)}(\chi)&=0\,,\\ n\,g_2^{(n-1)}(\chi)+(\chi-1)g_2^{(n)}(\chi)-g_3^{(n)}(\chi)&=0\,.\end{aligned}$$

These must clearly vanish for all $n\geq 4$ and solving these equations leads respectively to (6.8), (6.9).

[22] Notice that the A_0 term contributes to $G_1^{(2)}$ as $G_1^{(2)}\sim -A_0$ which is the trivial constant solution to the Ward identities.

[23] We may obtain no more independent constraints due to the recurrence relations mentioned in appendix B for the operators $D_n^{\pm}$.

To solve these recurrence relations it proves easiest to solve for $b_{\lambda_1\,\lambda_2}$ using (6.13) with (6.14), whence we may solve for $a_{\lambda_1\,\lambda_2}$ using (6.13) and $c_{\lambda_1\,\lambda_2}$ using (6.16). (6.15) demonstrates consistency. In this way the equation we get for $b_{\lambda_1\,\lambda_2}$ is given by,

$$\begin{aligned}&\frac{(\lambda_-+2\,\varepsilon-2)(\lambda_-+2\,\varepsilon-1)}{(\lambda_-+\varepsilon-2)(\lambda_-+\varepsilon-1)}\,(\lambda_1+\varepsilon-1)\,(\lambda_2+1)\,b_{\lambda_1-1\,\lambda_2+1}+\\&+\frac{(\lambda_-+1)(\lambda_-+2)}{(\lambda_-+\varepsilon+1)(\lambda_-+\varepsilon+2)}\,(\lambda_1+1)\,(\lambda_2-1)\,b_{\lambda_1+1\,\lambda_2-1}-\\&\quad-2\,\frac{\lambda_-(\lambda_-+2\,\varepsilon)+(2\,\varepsilon+1)(\varepsilon-1)}{(\lambda_-+\varepsilon-1)(\lambda_-+\varepsilon+1)}\,(\lambda_1+\varepsilon)\,\lambda_2\,b_{\lambda_1\,\lambda_2}=0\,.\end{aligned}\tag{6.17}$$

In fact we may use Jack polynomials as generating functions to solve (6.17) as follows,

$$\begin{aligned}\sum_{\lambda_1,\lambda_2}\Bigg(&\frac{(\lambda_-+2\,\varepsilon-2)(\lambda_-+2\,\varepsilon-1)}{(\lambda_-+\varepsilon-2)(\lambda_-+\varepsilon-1)}\,(\lambda_1+\varepsilon-1)\,(\lambda_2+1)\,b_{\lambda_1-1\,\lambda_2+1}+\\&+\frac{(\lambda_-+1)(\lambda_-+2)}{(\lambda_-+\varepsilon+1)(\lambda_-+\varepsilon+2)}\,(\lambda_1+1)\,(\lambda_2-1)\,b_{\lambda_1+1\,\lambda_2-1}-\\&\quad-2\,\frac{\lambda_-(\lambda_-+2\,\varepsilon)+(2\,\varepsilon+1)(\varepsilon-1)}{(\lambda_-+\varepsilon-1)(\lambda_-+\varepsilon+1)}\,(\lambda_1+\varepsilon)\,\lambda_2\,b_{\lambda_1\,\lambda_2}\Bigg)\,P^{(\varepsilon)}_{\lambda_1\,\lambda_2}(\chi,\chi')=\\&\quad(\chi+\chi'+2)(\chi+\chi'-2)\sum_{\lambda_1,\lambda_2}b_{\lambda_1\,\lambda_2}\,(\lambda_1+\varepsilon)\,\lambda_2\,P^{(\varepsilon)}_{\lambda_1-1\,\lambda_2-1}(\chi,\chi')=0\,,\end{aligned}\tag{6.18}$$

which implies that $b_{\lambda_1\,\lambda_2}$ is non-zero only for $\lambda_1=-\varepsilon$ or $\lambda_2=0$.[24]

It is not surprising that the restricted set of Jack polynomials $P^{(\varepsilon)}_{\lambda\,0}$ provide a basis for the $k=1$ solutions as

$$D_\varepsilon\,P^{(\varepsilon)}_{\lambda\,0}(\chi,\chi')=0\,,\tag{6.19}$$

which is the form of the integrability constraints (6.6).

Expanding the solution for $b_{\lambda_1\,\lambda_2}$ as

$$b_{\lambda_1\,\lambda_2}=\sum_\lambda\delta_{\lambda_1\lambda}\,\delta_{\lambda_2 0}\,b_\lambda\,,\tag{6.20}$$

then we may find, in the way mentioned,

$$\begin{aligned}a_{\lambda_1\,\lambda_2}&=\sum_\lambda\delta_{\lambda_1\lambda}\,\delta_{\lambda_2 0}\left(-b_\lambda+\frac{\lambda+1}{\lambda+2\varepsilon}\,b_{\lambda+1}\right),\\c_{\lambda_1\,\lambda_2}&=\sum_\lambda\delta_{\lambda_1\lambda}\,\delta_{\lambda_2 0}\left(\frac{\lambda+2\varepsilon-1}{\lambda}b_{\lambda-1}-b_\lambda\right).\end{aligned}\tag{6.21}$$

We may then reassemble the results to find that,

$$G^{(\varepsilon)}_1(\chi,\chi';\alpha,\alpha')=u^\varepsilon\,\mathcal{H}^{(\varepsilon)}_1(\chi,\chi';\alpha,\alpha')\,,\tag{6.22}$$

[24] As mentioned in appendix B we have that $P^{(\varepsilon)}_{\lambda_1\,\lambda_2}(\chi,\chi')=P^{(\varepsilon)}_{\lambda_2-\varepsilon\,\lambda_1+\varepsilon}(\chi,\chi')$ so that solutions of the form $P^{(\varepsilon)}_{-\varepsilon\,\lambda+\varepsilon}(\chi,\chi')=P^{(\varepsilon)}_{\lambda\,0}(\chi,\chi')$. However we may show that these are accounted for by the $\lambda_2=0$ possibility.

where,

$$\mathcal{H}_1^{(\varepsilon)}(\chi,\chi';\alpha,\alpha') = \sum_\lambda b_{1\,\lambda}\big(\tilde{P}^{(\varepsilon)}_{\lambda-1}(\chi,\chi') - (\alpha+\alpha')\tilde{P}^{(\varepsilon)}_{\lambda}(\chi,\chi') + \alpha\alpha'\,\tilde{P}^{(\varepsilon)}_{\lambda+1}(\chi,\chi')\big)\,, \qquad (6.23)$$

for,

$$b_{1\,\lambda} = \frac{\Gamma(\lambda+1)}{(2\varepsilon)_\lambda}\,b_\lambda\,. \qquad (6.24)$$

Using (5.20) we may rewrite this for integer ε as

$$G_1^{(\varepsilon)}(\chi,\chi';\alpha,\alpha')\big|_{\varepsilon=1,2,\dots} = u^\varepsilon(D_\varepsilon)^{\varepsilon-1}\bigg(\frac{(\chi\alpha-1)(\chi\alpha'-1)h(\chi)-(\chi'\alpha-1)(\chi'\alpha'-1)h(\chi')}{\chi-\chi'}\bigg), \qquad (6.25)$$

for some arbitrary single variable function $h(\chi)$. For $\alpha'=1$ in (6.25), this agrees with the functional form of (5.16). Similarly, for half integer ε, we may find $\tilde{P}^{(\varepsilon)}_\lambda(\chi,\chi') \propto (D_\varepsilon)^{\varepsilon-\frac{1}{2}}(\chi\chi')^{\frac{1}{2}\lambda}P_{\lambda+2\varepsilon-1}(s)$ for usual Legendre polynomials and with $s=(\chi+\chi')/(2\,u^{\frac{1}{2}})$ - this agrees with (5.11) for $h_j^{(\frac{1}{2})}(\chi,\chi') = \sum_\lambda a_\lambda(\chi\chi')^{\frac{1}{2}\lambda}P_{\lambda+2\varepsilon-1}(s)$, for some arbitrary a_λ.

6.1.2 Absorption of $\mathcal{H}^{(\varepsilon)}$ for SO(3) solutions

Here we prove a claim made in section 5.2 that $\mathcal{H}^{(\varepsilon)}$ of (5.37) with (5.18) may be absorbed into b_0 through a redefinition for $\varepsilon \neq 1$. In fact, this is a feature of the SO(n) case as well whereby the analysis here is also relevant - this is discussed in more detail for $k=2$ later. This claim is easily seen through an expansion in Jack polynomials whereby we may show that, for $\varepsilon \neq 1$,

$$\lim_{\mu\to 0} u^{2\varepsilon}\Delta_\varepsilon(\chi\alpha-1)(\chi'\alpha-1)(\Delta_\varepsilon)^{-1}\sum_\lambda a_\lambda\frac{(2\varepsilon)_{\lambda-1}}{\Gamma(\lambda)}P^{(\varepsilon)}_{\lambda+\mu-\varepsilon-1\,\mu-\varepsilon}(\chi,\chi') = u^\varepsilon\mathcal{H}^{(\varepsilon)}(\chi,\chi';\alpha)\,, \qquad (6.26)$$

with the operator Δ_ε defined in (5.26) and for which the following holds, namely,

$$\Delta_f\,P^{(\varepsilon)}_{\lambda_1\,\lambda_2}(\chi,\chi') = \mathcal{E}^{(f)}_{\lambda_1\,\lambda_2}\,P^{(\varepsilon)}_{\lambda_1\,\lambda_2}(\chi,\chi')\,, \qquad \mathcal{E}^{(f)}_{\lambda_1\,\lambda_2} = (\lambda_1+1+\varepsilon)_{f-1}(\lambda_2+1)_{f-1}\,, \qquad (6.27)$$

following from results in appendix B. Hence by making the redefinition,

$$b_0 \to b_0 - (\Delta_\varepsilon)^{-1}\sum_\lambda a_\lambda\,\frac{(2\varepsilon)_{\lambda-1}}{\Gamma(\lambda)}\,P^{(\varepsilon)}_{\lambda+\mu-\varepsilon-1\,\mu-\varepsilon}(\chi,\chi')\,, \qquad (6.28)$$

and subsequently taking the limit $\mu\to 0$ we may absorb the $\mathcal{H}^{(\varepsilon)}$ contribution. The limit is taken in this way because the modification in (6.28) has an implicit factor coming from $(\Delta_\varepsilon)^{-1}$, whose action on the Jack polynomials gives $(\mathcal{E}^{(\varepsilon)}_{\lambda+\mu-\varepsilon-1\,\mu-\varepsilon})^{-1}$, which has poles for $\mu=0$ and $\varepsilon\neq 2,3,\dots$.

For $\varepsilon=2$ we may reproduce (5.31), (5.32) and (5.33), whereby defining

$$\sum_\lambda a_\lambda\tilde{P}^{(2)}_{\lambda-1} = -D_2\sum_\lambda a_\lambda\frac{\chi^{\lambda+2}-\chi'^{\lambda+2}}{\chi-\chi'} = -D_2\left(\frac{h(\chi)-h(\chi')}{\chi-\chi'}\right), \qquad (6.29)$$

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so that

$$\mathcal{H}^{(2)}(\chi,\chi';\alpha) = D_2\left(\frac{(\chi\alpha-1)h(\chi)-(\chi'\alpha-1)h(\chi')}{\chi-\chi'}\right), \tag{6.30}$$

then the modification from (6.28) is given by

$$\frac{1}{u^2}\sum_\lambda a_\lambda(\mathcal{E}^{(2)}_{\lambda-3\ -2})^{-1}\tilde{P}^{(2)}_{\lambda-1} = \frac{1}{u^2}D_2\sum_\lambda \frac{a_\lambda}{\lambda}\frac{\chi^{\lambda+2}-\chi'^{\lambda+2}}{\chi-\chi'} = -\frac{1}{u^2}D_2\left(\frac{g(\chi)-g(\chi')}{\chi-\chi'}\right), \tag{6.31}$$

for which g satisfies (5.33) in terms of h.

6.1.3 Proof of (5.28) for general ε

Much use was made of the operator identity (5.28) in finding solutions to the Ward identities for SO(3) R symmetry. Indeed this identity is relevant for the SO(n) R symmetry case also. Turning to the proof of (5.28), we wish to show that, for arbitrary $f(\chi,\chi')$,

$$\frac{1}{\chi^2}u^{2\varepsilon}\partial_\chi f(\chi,\chi') = \frac{1}{x}u^\varepsilon\partial_\chi u^\varepsilon\Delta_\varepsilon(\chi+\chi')u^{-1}(\Delta_\varepsilon)^{-1}f(\chi,\chi') - \partial_\chi u^{2\varepsilon}\Delta_\varepsilon u^{-1}(\Delta_\varepsilon)^{-1}f(\chi,\chi')\,. \tag{6.32}$$

To prove this we consider $f(\chi,\chi')$, $g(\chi,\chi')$, $h(\chi,\chi')$ where

$$f=\sum_{\lambda_1,\lambda_2} f_{\lambda_1\,\lambda_2}P^{(\varepsilon)}_{\lambda_1\,\lambda_2}\,,\qquad g=\sum_{\lambda_1,\lambda_2} g_{\lambda_1\,\lambda_2}P^{(\varepsilon)}_{\lambda_1\,\lambda_2}\,,\qquad h=\sum_{\lambda_1,\lambda_2} h_{\lambda_1\,\lambda_2}P^{(\varepsilon)}_{\lambda_1\,\lambda_2}\,, \tag{6.33}$$

and attempt to find conditions on g, h coming from

$$\frac{1}{\chi^2}u^{2\varepsilon}\partial_\chi f(\chi,\chi') = \frac{1}{\chi}u^\varepsilon\partial_\chi u^\varepsilon g(\chi,\chi') + \partial_\chi u^{2\varepsilon}h(\chi,\chi')\,. \tag{6.34}$$

From (6.34) with (6.33) we may obtain two independent recurrence relations determining $g_{\lambda_1\,\lambda_2}$, $h_{\lambda_1\,\lambda_2}$ in terms of $f_{\lambda_1\,\lambda_2}$. Defining

$$h_{\lambda_1\,\lambda_2} = -\frac{(\lambda_1+\varepsilon+1)(\lambda_2+1)}{(\lambda_1+2\varepsilon)(\lambda_2+\varepsilon)}\,\tilde{h}_{\lambda_1+1\,\lambda_2+1}\,, \tag{6.35}$$

we may obtain a recurrence relation from these of the form (6.17) with

$$b_{\lambda_1\,\lambda_2}\to f_{\lambda_1\,\lambda_2}-\tilde{h}_{\lambda_1\,\lambda_2}\,, \tag{6.36}$$

determining $h_{\lambda_1\,\lambda_2}$ in terms of $f_{\lambda_1\,\lambda_2}$. Similarly we may find that

$$\begin{aligned}(\lambda_1+\lambda_2+2\varepsilon)g_{\lambda_1\,\lambda_2} = {}& \frac{(\lambda_1+\varepsilon+1)(\lambda_-+1)}{\lambda_-+\varepsilon+1}\left(f_{\lambda_1+1\,\lambda_2}+\frac{\lambda_2}{\lambda_1+2\varepsilon}\tilde{h}_{\lambda_1+1\,\lambda_2}\right)+\\ &+\frac{(\lambda_2+1)(\lambda_-+2\varepsilon-1)}{\lambda_-+\varepsilon-1}\left(f_{\lambda_1\,\lambda_2+1}+\frac{\lambda_1+\varepsilon}{\lambda_2+\varepsilon}\tilde{h}_{\lambda_1\,\lambda_2+1}\right).\end{aligned} \tag{6.37}$$

We therefore obtain that $\tilde{h}_{\lambda_1\,\lambda_2} = f_{\lambda_1\,\lambda_2}+\sum_\lambda h_\lambda\,\delta_{\lambda_1,\lambda}\delta_{\lambda_2,0}$ so that,

$$h_{\lambda_1\,\lambda_2} = -\frac{(\lambda_1+\varepsilon+1)(\lambda_2+1)}{(\lambda_1+2\varepsilon)(\lambda_2+\varepsilon)}\,f_{\lambda_1+1\,\lambda_2+1}\,, \tag{6.38}$$

and

$$g_{\lambda_1\,\lambda_2} = \frac{(\lambda_1+\varepsilon+1)(\lambda_-+1)}{(\lambda_1+2\varepsilon)(\lambda_-+\varepsilon+1)} f_{\lambda_1+1\,\lambda_2} + \frac{(\lambda_2+1)(\lambda_-+2\varepsilon-1)}{(\lambda_2+\varepsilon)(\lambda_-+\varepsilon-1)} f_{\lambda_1\,\lambda_2+1}\,. \tag{6.39}$$

With (6.33) we therefore obtain using (6.27)

$$h(\chi,\chi') = -\Delta_\varepsilon\, u^{-1}\,(\Delta_\varepsilon)^{-1} f(\chi,\chi')\,, \qquad g(\chi,\chi') = \Delta_\varepsilon(\chi+\chi')u^{-1}(\Delta_\varepsilon)^{-1} f(\chi,\chi')\,, \tag{6.40}$$

proving (6.32) and hence (5.28).

6.2 The general k case

In this section we solve the Ward identities (4.31) for general k, corresponding to the case when the four-point function has operators belonging to the rank k symmetric traceless representation of the $\mathrm{SO}(n)$ R symmetry group. We start with the simplest physical case — the $k = 2$ one — whereby we make further use of the operator identity (5.28), along with certain extensions of it, to solve the Ward identities in a manner not unlike the $k = 2$ case considered in section 5.2. The difference is that in contrast to the SO(3) case, where we could reduce the equations to $k = 1$ ones, here the reduced equations are not of the $k = 1$ form for $\mathrm{SO}(n)$. Nevertheless, the analysis for $k = 1$ of $\mathrm{SO}(n)$ is important here because it implies that the solutions to the reduced equations are expressible in terms of expansions in the restricted set of Jack polynomials $\tilde P^{(\varepsilon)}_\lambda(\chi,\chi')$. This simplifies the analysis considerably.

6.2.1 The $k = 2$ case

We start with the $k = 2$ case for which we have the following three independent equations following from the Ward identities (4.31), namely,

$$\sum_{i=0}^{2} \frac{1}{\chi^i} u^{i\varepsilon}\,\partial_\chi\, a_{i0} = 0\,, \qquad \sum_{\substack{i,j\geq 0\\ i+j=2}} (1-\chi)^j\, v^{-j\varepsilon}\,\partial_\chi\, a_{ij} = 0\,, \qquad \sum_{i=0}^{2} \Big(\frac{\chi-1}{\chi}\Big)^i \Big(\frac{u}{v}\Big)^{i\varepsilon}\,\partial_\chi\, a_{0i} = 0\,, \tag{6.41}$$

along with three more obtained from symmetry under $\chi \leftrightarrow \chi'$.

To solve these we employ two further operator identities in addition to (5.28), namely,

$$\begin{aligned}
\frac{1}{(1-\chi)^j}\, v^{j\varepsilon}\,\partial_\chi\,(D_\varepsilon)^{\varepsilon-1}\, v \;&=\; \frac{1}{(1-\chi)^{j-1}} v^{(j-1)\varepsilon}\,\partial_\chi\, v^\varepsilon\,(D_\varepsilon)^{\varepsilon-1}(2-\chi-\chi') - \\
&\quad - \frac{1}{(1-\chi)^{j-2}} v^{(j-2)\varepsilon}\,\partial_\chi\, v^{2\varepsilon}(D_\varepsilon)^{\varepsilon-1}\,, \\
\Big(\frac{\chi-1}{\chi}\Big)^j \Big(\frac{u}{v}\Big)^{j\varepsilon} \partial_\chi\, v^{2\varepsilon}(D_\varepsilon)^{\varepsilon-1}\, u \;&=\; \Big(\frac{\chi-1}{\chi}\Big)^{j-1} \Big(\frac{u}{v}\Big)^{(j-1)\varepsilon}\,\partial_\chi\,(uv)^\varepsilon\,(D_\varepsilon)^{\varepsilon-1}(2u-\chi-\chi') - \\
&\quad - \Big(\frac{\chi-1}{\chi}\Big)^{j-2} \Big(\frac{u}{v}\Big)^{(j-2)\varepsilon}\,\partial_\chi\, u^{2\varepsilon}\,(D_\varepsilon)^{\varepsilon-1} v\,,
\end{aligned} \tag{6.42}$$

although these may be obtained from (5.28); the first from symmetry under $\chi\to 1-\chi$, $\chi'\to 1-\chi'$ and the second from symmetry under $\chi\to\chi/(\chi-1)$, $\chi'\to\chi'/(\chi'-1)$. The second case may be seen using

$$D_\varepsilon|_{u\to 1/u,\, v\to v/u} = u^{\varepsilon+2}\, D_\varepsilon\, u^{-\varepsilon}\,, \tag{6.43}$$

along with[25]

$$(D_\varepsilon|_{u\to 1/u,\, v\to v/u})^{\varepsilon-1} = u^{2\varepsilon}(D_\varepsilon)^{\varepsilon-1}u^{-2} \quad \Rightarrow \quad (D_\varepsilon|_{u\to u/v,\, v\to 1/v})^{\varepsilon-1} = v^{2\varepsilon}(D_\varepsilon)^{\varepsilon-1}v^{-2}\,. \tag{6.45}$$

Examining the second two equations in (6.41) and using (6.42) we may, by choosing,

$$a_{02}(\chi,\chi') = v^{2\varepsilon}\,\Delta_\varepsilon\, u\, a(\chi,\chi')\,, \tag{6.46}$$

(where $a(\chi,\chi') = a(\chi',\chi)$ is general) reduce them to the following two equations, namely,

$$\begin{aligned}\partial_\chi\big(a_{00} - u^{2\varepsilon}\Delta_\varepsilon v a\big) + \frac{\chi-1}{\chi}\Big(\frac{u}{v}\Big)^\varepsilon \partial_\chi\big(a_{01} + (u\,v)^\varepsilon\Delta_\varepsilon(2u-\chi-\chi')a\big) &= 0\,,\\ \partial_\chi\big(a_{11} + v^\varepsilon\Delta_\varepsilon\, u(2-\chi-\chi')a\big) + \frac{1}{1-\chi}v^\varepsilon\partial_\chi\big(a_{20} - \Delta_\varepsilon u\, v\, a\big) &= 0\,,\end{aligned} \tag{6.47}$$

(along with two similar obtained by $\chi \leftrightarrow \chi'$) which is *almost* of the $k=1$ form (6.2). In fact what the $k=1$ analysis of the last section implies is that the solutions to (6.47) are expandable without loss of generality in terms of the restricted set of Jack polynomials $P^{(\varepsilon)}_{\lambda\,0}(\chi,\chi') \propto \tilde{P}^{(\varepsilon)}_{\lambda}(\chi,\chi')$.

For $\varepsilon = 2$ we may solve these equations directly using the analysis of the last section. We may find that

$$\begin{aligned} a_{00} &= u^4\Delta_2\, v\, a + u^2 D_2\bigg(\frac{g_1(\chi)-g_1(\chi')}{\chi-\chi'}\bigg)\,,\\ a_{01} &= (uv)^2\,\Delta_2(\chi+\chi'-2u)a + v^2 D_2\bigg(\frac{g_2(\chi)-g_2(\chi')}{\chi-\chi'}\bigg)\,,\\ a_{11} &= v^2\Delta_2 u(\chi+\chi'-2)a + v^2 D_2\bigg(\frac{g_3(\chi)-g_3(\chi')}{\chi-\chi'}\bigg)\,,\\ a_{20} &= \Delta_2\, u\, v\, a + D_2\bigg(\frac{g_4(\chi)-g_4(\chi')}{\chi-\chi'}\bigg)\,, \end{aligned} \tag{6.48}$$

where

$$\begin{aligned} g_1(\chi) &= -\frac{\chi-1}{\chi}g_2(\chi) + A_0\frac{1}{\chi} + A_1 + A_2\chi + A_3\chi^2\,,\\ g_4(\chi) &= (\chi-1)g_3(\chi) + B_1 + B_2\chi + B_3\chi^2 + B_4\chi^4\,, \end{aligned} \tag{6.49}$$

with A_i, B_i being arbitrary constants. Similarly solving the first equation in (6.41) and using (5.28) we find that,

$$a_{10} = -u^2\Delta_2\, v(\chi+\chi')a + D_2\bigg(\frac{g_5(\chi)-g_5(\chi')}{\chi-\chi'}\bigg) + u^2 D_2\bigg(\frac{g_6(\chi)-g_6(\chi')}{\chi-\chi'}\bigg)\,, \tag{6.50}$$

[25]We may see this quite easily from Jack polynomials with the operator $\Delta_\varepsilon = (D_\varepsilon)^{\varepsilon-1}u^{\varepsilon-1}$. We may show that

$$\begin{aligned}\Delta_\varepsilon|_{u\to 1/u,\, v\to v/u}P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi') &= \Delta_\varepsilon|_{u\to 1/u,\, v\to v/u}P^{(\varepsilon)}_{-\lambda_2\,-\lambda_1}(1/\chi,1/\chi')\\ &= u^{2\varepsilon}\Delta_\varepsilon\, u^{-2\varepsilon}P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi') \qquad \text{for } \varepsilon = 0,\tfrac{1}{2},1,\tfrac{3}{2},\ldots\,, \quad \lambda_- = 0,1,2\ldots\,,\end{aligned}$$

so that

$$\Delta_\varepsilon|_{u\to 1/u,\, v\to v/u} = u^{2\varepsilon}\Delta_\varepsilon\, u^{-2\varepsilon} \quad \Rightarrow \quad \Delta_\varepsilon|_{u\to u/v,\, v\to 1/v} = v^{2\varepsilon}\Delta_\varepsilon\, v^{-\varepsilon-1}\,.$$

where

$$\begin{aligned} g_5(\chi) &= (\chi-1)g_2(\chi) + B'_1 + B'_2\chi + B'_3\chi^2 + B'_4\chi^3\,, \\ g_6(\chi) &= -\frac{\chi-1}{\chi}g_3(\chi) + A'_0\frac{1}{\chi} + A'_1 + A'_2\chi + A'_3\chi^2\,, \end{aligned} \tag{6.51}$$

with A'_i, B'_i being arbitrary constants. With these expressions for a_{ij} we may write the full solution to the Ward identities for the $\varepsilon = 2$ case as

$$\begin{aligned} G_2^{(2)}(\chi,\chi';\alpha,\alpha') &= u^4\Delta_2\,(\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)a(\chi,\chi') \\ &\quad + u^2\mathcal{H}_1^{(2)}(\chi,\chi';\alpha,\alpha') + u^4\alpha\alpha'\,\mathcal{H}_2^{(2)}(\chi,\chi';\alpha,\alpha')\,, \end{aligned} \tag{6.52}$$

where

$$\mathcal{H}_i^{(2)}(\chi,\chi';\alpha,\alpha') = D_2\bigg(\frac{(\chi\alpha-1)(\chi\alpha'-1)h_i(\chi) - (\chi'\alpha-1)(\chi'\alpha'-1)h_i(\chi')}{\chi-\chi'}\bigg)\,, \tag{6.53}$$

for

$$h_1(\chi) = \frac{1}{\chi}g_2(\chi) + A_0\frac{1}{\chi} + B'_4\chi\,, \qquad h_2(\chi) = \frac{1}{\chi}g_3(\chi) + A'_0\frac{1}{\chi} + B_4\chi\,, \tag{6.54}$$

being arbitrary single variable functions.

Using expansions in terms of $P_{\lambda\,0}^{(\varepsilon)}(\chi,\chi')$, then for general ε we may relatively easily find that, for arbitrary $b_{1\,\lambda}$, $b_{2\,\lambda}$,

$$\begin{aligned} a_{00} &= u^{2\varepsilon}\Delta_\varepsilon\, v\, a + u^\varepsilon \sum_\lambda \big(-b_{1\,\lambda} + b_{1\,\lambda+1}\big)\tilde{P}_\lambda^{(\varepsilon)}\,, \\ a_{01} &= (uv)^\varepsilon\,\Delta_\varepsilon(\chi+\chi'-2u)a + v^\varepsilon\sum_\lambda b_{1\,\lambda}\tilde{P}_\lambda^{(\varepsilon)}\,, \\ a_{11} &= v^\varepsilon\Delta_\varepsilon u(\chi+\chi'-2)a + v^\varepsilon\sum_\lambda b_{2\,\lambda}\tilde{P}_\lambda^{(\varepsilon)}\,, \\ a_{20} &= \Delta_\varepsilon\, u\, v\, a + \sum_\lambda \big(b_{2\,\lambda-1} - b_{2\,\lambda}\big)\tilde{P}_\lambda^{(\varepsilon)}\,, \end{aligned} \tag{6.55}$$

then from the first equation in (6.41) and using (5.28) we may find,

$$a_{10} = -u^\varepsilon\Delta_\varepsilon\, v(\chi+\chi')a + \sum_\lambda \big(b_{1\,\lambda-1} - b_{1\,\lambda}\big)\tilde{P}_\lambda^{(\varepsilon)} + u^\varepsilon\sum_\lambda\big(-b_{2\,\lambda} + b_{2\,\lambda+1}\big)\tilde{P}_\lambda^{(\varepsilon)}\,. \tag{6.56}$$

Taking account of these solutions then we may easily reassemble them to find that

$$\begin{aligned} G_2^{(\varepsilon)}(\chi,\chi';\alpha,\alpha') &= u^{2\varepsilon}\Delta_\varepsilon\,(\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)a(\chi,\chi') + \\ &\quad + u^\varepsilon\mathcal{H}_1^{(\varepsilon)}(\chi,\chi';\alpha,\alpha') + u^{2\varepsilon}\alpha\alpha'\,\mathcal{H}_2^{(\varepsilon)}(\chi,\chi';\alpha,\alpha')\,, \end{aligned} \tag{6.57}$$

where $\mathcal{H}_i^{(\varepsilon)}(\chi,\chi';\alpha,\alpha')$ is given by (6.23) with $b_{1\,\lambda} \to b_{i\,\lambda}$.

Due to the arbitrary freedom we have in $a(\chi,\chi')$, we may re-define

$$a(\chi,\chi') \to a(\chi,\chi') - u^{-2}\,(\Delta_\varepsilon)^{-1}\sum_\lambda b_{2\,\lambda}\,\tilde{P}_{\lambda+1}^{(\varepsilon)}(\chi,\chi')\,, \tag{6.58}$$

whereby using that for $p=0,1,2$,

$$\Delta_\varepsilon\Big(\frac{\chi+\chi'}{u}\Big)^p\,\tilde P^{(\varepsilon)}_{\lambda}(\chi,\chi')=\mathcal{E}^{(\varepsilon)}_{\lambda\,0}\,\tilde P^{(\varepsilon)}_{\lambda-p}(\chi,\chi')\,,\tag{6.59}$$

we may dispose of $b_{2\,\lambda}$ contributions in (6.57) for $\varepsilon\neq1,2$. For $\varepsilon=2$, those left over may be easily computed to be given by

$$G^{(2)}_2(\chi,\chi';\alpha,\alpha')\sim u^2\sum_\lambda b_{2\,\lambda-2}\,\frac{\lambda}{\lambda+2}\big(\tilde P^{(\varepsilon)}_{\lambda-1}(\chi,\chi')-(\alpha+\alpha')\tilde P^{(\varepsilon)}_{\lambda}(\chi,\chi')+\alpha\alpha'\,\tilde P^{(\varepsilon)}_{\lambda+1}(\chi,\chi')\big)\,,\tag{6.60}$$

so that by re-defining

$$b_{1\,\lambda}\to b_{1\,\lambda}-\frac{\lambda}{\lambda+2}b_{2\,\lambda-2}\,,\tag{6.61}$$

we may absorb $b_{2\,\lambda}$ into $b_{1\,\lambda}$ contributions in this case as well.[26]

In the case $\varepsilon=2$, a more direct way of seeing absorbtion for one of $\mathcal{H}^{(2)}_1$, $\mathcal{H}^{(2)}_2$ is through use of the following identity, namely,

$$\begin{aligned}(\chi\chi')^4D_2\,(\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)\,\frac{1}{\chi\chi'}\,D_2\bigg(\frac{h(\chi)-h(\chi')}{\chi-\chi'}\bigg)=\\ =(\chi\chi')^2\mathcal{H}^{(2)}_1(\chi,\chi';\alpha,\alpha')+\alpha\alpha'(\chi\chi')^4\mathcal{H}^{(2)}_2(\chi,\chi';\alpha,\alpha')\,,\end{aligned}\tag{6.62}$$

for

$$h_1(\chi)=-2h(\chi)+\chi h'(\chi)\,,\qquad h_2(\chi)=-\frac{1}{\chi}h'(\chi)\,.\tag{6.63}$$

Thus we may absorb $\mathcal{H}^{(2)}_2$ via modifications of the unrestricted two-variable part and of $\mathcal{H}^{(2)}_1$, which is essentially the point made above using expansions.

We thus have that,

$$\begin{aligned}G^{(\varepsilon)}_2(\chi,\chi';\alpha,\alpha')\big|_{\varepsilon\neq1}=u^{2\varepsilon}\Delta_\varepsilon\,(\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)a(\chi,\chi')+\\ +u^\varepsilon\,\mathcal{H}^{(\varepsilon)}_1(\chi,\chi';\alpha,\alpha')\,,\end{aligned}\tag{6.64}$$

whereby for $\varepsilon\neq2$ we may absorb $\mathcal{H}^{(\varepsilon)}_1$ through redefinitions of $a(\chi,\chi')$.[27]

Further absorbtion of the $\mathcal{H}^{(2)}_1$ contributions for $\varepsilon=2$ is not possible by the modification $a(\chi,\chi')\to a(\chi,\chi')+f(\chi,\chi')/u$. Indeed, $\mathcal{H}^{(2)}_1$ contains three out of the six monomials in the polynomial in α,α'. Making a modification of $a(\chi,\chi')$ which might absorb these

[26] We see later, when performing the conformal partial wave expansion, that such re-absorption of $b_{2\,\lambda}$ contributions is not possible for $\varepsilon=1$.

[27] We make the same choice of modification of the two variable function as before in (6.26): we may find that

$$\begin{aligned}\lim_{\mu\to0}u^{2\varepsilon}\Delta_\varepsilon(\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)(\Delta_\varepsilon)^{-1}\sum_\lambda b_{1\lambda}\frac{(2\varepsilon)_{\lambda-1}}{\Gamma(\lambda)}\,P^{(\varepsilon)}_{\lambda+\mu-\varepsilon-1\,\mu-\varepsilon}(\chi,\chi')=\\ =u^\varepsilon\,\mathcal{H}_1(\chi,\chi';\alpha,\alpha')\,.\end{aligned}$$

three terms, but which should not affect the remaining three, we must require that

$$D_2\, u^2 f = 0\,, \qquad D_2 u(\chi+\chi')f = 0\,, \qquad D_2(\chi+\chi')^2 f = 0\,. \tag{6.65}$$

These equations are incompatible, as is easily seen by using the first of them in the other two and remembering that $f(\chi,\chi')$ must be a symmetric function.

However, in the case $\varepsilon = 2$ there is a different mechanism of absorption. Using the identity

$$\begin{aligned} &D_2\Bigg((\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)\left(\frac{h(\chi)-h(\chi')}{(\chi-\chi')^3}\right)\Bigg) - \\ &-\alpha\alpha' D_2\left(\frac{(\chi\alpha-1)(\chi\alpha'-1)h(x)-(\chi'\alpha-1)(\chi'\alpha'-1)h(\chi')}{\chi-\chi'}\right) = \\ &\quad = \frac{1}{(\chi-\chi')^4}\left((\chi\alpha-1)^2(\chi\alpha'-1)^2 h'(\chi)+(\chi'\alpha-1)^2(\chi'\alpha'-1)^2 h'(\chi')\right), \end{aligned} \tag{6.66}$$

we can rewrite (6.64) as follows:

$$\begin{aligned} G_2^{(2)}(\chi,\chi';\alpha,\alpha') = &-\frac{(\chi\chi')^4}{(\chi-\chi')^4}\,\Delta\Big((\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)F(\chi,\chi')\Big) - \\ &-\frac{1}{(\chi-\chi')^4}\left((\chi\alpha-1)^2(\chi\alpha'-1)^2F(\chi,\chi)+\right. \\ &\left.+(\chi'\alpha-1)^2(\chi'\alpha'-1)^2F(\chi',\chi')\right), \end{aligned} \tag{6.67}$$

where we have used the differential operator Δ defined by the relation

$$D_2\,\frac{1}{(\chi-\chi')^2}f(\chi,\chi') = -\frac{1}{(\chi-\chi')^4}\,\Delta\, f(\chi,\chi')\,. \tag{6.68}$$

The new function $F(\chi,\chi')$ is obtained from $a(\chi,\chi')$ and $h(\chi)$ in the following way:

$$F(\chi,\chi') = (\chi-\chi')^2 a(\chi,\chi') + \frac{h(\chi)-h(\chi')}{\chi-\chi'}\,. \tag{6.69}$$

Thus, assuming regularity of the function $a(\chi,\chi')$ as $\chi\to\chi'$, the single variable function $h(\chi)$ is identified with $F(\chi,\chi)$,

$$F(\chi,\chi) = h'(\chi)\,. \tag{6.70}$$

We conclude that in this case the set of two functions $a(\chi,\chi')$ and $h(\chi)$ has effectively been reduced to the *single function of two variables* $F(\chi,\chi')$. The form (6.67) first appeared in [39].

We remark that both forms of the amplitude, (6.64) and (6.67), are regular in the limit $\chi\to\chi'$ under the assumption that the functions $a(\chi,\chi')$ and $h(\chi)$ are differentiable.

6.2.2 The general k case

A similar analysis as that done for $k = 2$ above becomes cumbersome, to say that least, for higher k. Indeed we can see this already for the $k = 3$ case which is considered in appendix C. A way out of this difficulty is to find a sufficiently arbitrary solution to the Ward identities (4.31) whose freedom may then be used to reduce the constraining equations on the correlation function to a set of simpler ones. This is what we do next.

As a partial solution to the Ward identities (4.31) it seems natural to consider

$$\begin{aligned}\mathcal{F}_k^{(\varepsilon)}(\chi,\chi';\alpha,\alpha') = & \sum_{0\leq n+m\leq k-2} G_{nm}(\chi,\chi')(\alpha\alpha')^n\big((\alpha-1)(\alpha'-1)\big)^m \times \\ & \times\Delta_\varepsilon(\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)\mathcal{F}_{nm}(\chi,\chi')\,, \qquad (6.71)\end{aligned}$$

where G_{nm}, $\mathcal{F}_{nm}$ are symmetric in χ,χ' and the restriction $m+n\leq k-2$ follows as the maximum degree of $G_k^{(\varepsilon)}$ in α,α' is k.

The constraints on this, if it is to satisfy the Ward identities for general $\mathcal{F}_{mn}$, are greatly simplified if we first prove that,

$$\chi\frac{\partial}{\partial\chi}\Delta_\varepsilon(\chi\alpha-1)(\chi'\alpha-1)f(\chi,\chi')\Big|_{\alpha\to1/\chi} = \varepsilon\Delta_\varepsilon(\alpha(\chi+\chi')-2)f(\chi,\chi')\Big|_{\alpha\to1/\chi}. \qquad (6.72)$$

With the identity (6.72), then the Ward identities (4.31), with (6.71), simply imply that,

$$\Big((\chi-1)\chi\frac{\partial}{\partial\chi}+\varepsilon(m-(n+2)(\chi-1))\Big)G_{nm}(\chi,\chi')=0\,, \qquad (6.73)$$

which may be simply solved by, taking account of χ,χ' symmetry,

$$G_{nm}(\chi,\chi')\propto u^{(2+m+n)\varepsilon}v^{-m\varepsilon}\,. \qquad (6.74)$$

Due to the simple action of Δ_ε on Jack polynomials, perhaps the simplest way of proving (6.72) is to first expand,

$$f(\chi,\chi')=(\Delta_\varepsilon)^{-1}\sum_{\lambda_1,\lambda_2}a_{\lambda_1\lambda_2}P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi')\,, \qquad (6.75)$$

for arbitrary $a_{\lambda_1\lambda_2}$. By using the identity (from results in appendix B)

$$\begin{aligned}\chi\frac{\partial}{\partial\chi}P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi') = & \frac{1}{2}(\lambda_1+\lambda_2)P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi')+ \\ & +\frac{1}{\chi-\chi'}\frac{\lambda_-(\lambda_-+2\varepsilon)}{2(\lambda_-+\varepsilon)}\big(P^{(\varepsilon)}_{\lambda_1+1\,\lambda_2}(\chi,\chi')-P^{(\varepsilon)}_{\lambda_1\,\lambda_2+1}(\chi,\chi')\big)\,, \qquad (6.76)\end{aligned}$$

along with repeated application of $(\chi+\chi')P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi')$ as in appendix B, we may determine that, with (6.75),

$$\chi\frac{\partial}{\partial\chi}\Delta_\varepsilon(\chi\alpha-1)(\chi'\alpha-1)f(\chi,\chi')\Big|_{\alpha\to1/\chi}=\frac{1}{\chi}\left(f_+(\chi,\chi')+f_-(\chi,\chi')\right), \qquad (6.77)$$

where,

$$f_+(\chi,\chi')=\varepsilon(\varepsilon-1)\sum_{\lambda_1,\lambda_2}\left(a_{\lambda_1-1\,\lambda_2}\frac{1}{\lambda_1+\varepsilon}+a_{\lambda_1\,\lambda_2-1}\frac{1}{\lambda_2}\right)P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi')\,, \qquad (6.78)$$

being symmetric in χ, χ', and

$$
\begin{aligned}
f_-(\chi,\chi') = -\varepsilon \frac{1}{\chi-\chi'} \sum_{\lambda_1,\lambda_2} \Bigg(& a_{\lambda_1-1\,\lambda_2} \frac{(\lambda_-+2\varepsilon-1)(\lambda_-+2\varepsilon-2)}{(\lambda_-+\varepsilon-1)(\lambda_-+\varepsilon-2)} - \\
& - 2a_{\lambda_1-1\,\lambda_2-1} \frac{\lambda_-(\lambda_-+2\varepsilon)+(\varepsilon-1)(2\varepsilon+1)}{(\lambda_-+\varepsilon-1)(\lambda_-+\varepsilon+1)} + \\
& + a_{\lambda_1\,\lambda_2-2} \frac{(\lambda_-+1)(\lambda_-+2)}{(\lambda_-+\varepsilon+1)(\lambda_-+\varepsilon+2)} \Bigg) P^{(\varepsilon)}_{\lambda_1\,\lambda_2}(\chi,\chi')\,, \quad (6.79)
\end{aligned}
$$

being anti-symmetric in χ, χ'. It is not then difficult to show that (6.77) with (6.78) and (6.79) coincides with the right hand side of (6.72), whereby we may determine that,

$$
\begin{aligned}
f_+(\chi,\chi') &= \varepsilon\Delta_\varepsilon(\chi+\chi') f(\chi,\chi') - \varepsilon \frac{\chi+\chi'}{u} \Delta_\varepsilon f(\chi,\chi')\,, \\
f_-(\chi,\chi') &= \varepsilon \frac{\chi-\chi'}{u} \Delta_\varepsilon f(\chi,\chi')\,. \qquad (6.80)
\end{aligned}
$$

In summary the partial solution we have found to the Ward identities (4.31) is given by,

$$
\begin{aligned}
\mathcal{F}^{(\varepsilon)}_k(\chi,\chi';\alpha,\alpha') = \sum_{0\le n+m\le k-2} & u^{(m+n+2)\varepsilon}\, v^{-m\varepsilon}\, (\alpha\alpha')^n \left((\alpha-1)(\alpha'-1)\right)^m \times \\
& \times \Delta_\varepsilon (\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)\mathcal{F}_{nm}(\chi,\chi')\,, \qquad (6.81)
\end{aligned}
$$

in terms of $\frac{1}{2}k(k-1)$ arbitrary two-variable functions $\mathcal{F}_{nm}(\chi,\chi')$. For $k = 2$, (6.81) clearly coincides with (6.64). When $\varepsilon = 1$, then $\Delta_\varepsilon \to 1$ and the solution coincides with two-variable part of the four-dimensional solution obtained earlier.

We may use (6.81) to find the solution to the Ward identities (4.31) more generally. To show this we use the arbitrariness in $\mathcal{F}_{nm}(\chi,\chi')$ to cancel all a_{nm}, $m \ge 2$, in the original four point function $G^{(\varepsilon)}_k(\chi,\chi';\alpha,\alpha')$ so that[28]

$$
\begin{aligned}
& G^{(\varepsilon)}_k(\chi,\chi';\alpha,\alpha') - \mathcal{F}^{(\varepsilon)}_k(\chi,\chi';\alpha,\alpha') = \\
& \quad = \sum_{n=0}^{k} u^{n\varepsilon}(\alpha\alpha')^n\, a'_{n0}(\chi,\chi') + \sum_{n=0}^{k-1} u^{n\varepsilon}\left(\frac{u}{v}\right)^{\varepsilon} (\alpha\alpha')^n (\alpha-1)(\alpha'-1)\, a'_{n1}(\chi,\chi')\,, \quad (6.82)
\end{aligned}
$$

for some symmetric $a'_{n0}(\chi,\chi')\,, a'_{n1}(\chi,\chi')$. The left-hand side of (6.82) obviously satisfies the Ward identities (4.31) and this places constraints on $a'_{n0}\,, a'_{n1}$. These constraint equations

[28] This is certainly possible as there are $\frac{1}{2}k(k-1)$ a_{nm}, $m \ge 2$, in $G^{(\varepsilon)}_k(\chi,\chi';\alpha,\alpha')$ - in fact there is some freedom left over in $\mathcal{F}_{mn}(\chi,\chi')$ due to Δ_ε generally having a non-trivial kernel, which has consequences for the restricted solutions we obtain.

following from the Ward identities are given by

$$\begin{aligned}
\frac{1}{1-\chi}v^{\varepsilon}\,\partial_\chi a'_{00} - \frac{1}{\chi}u^{\varepsilon}\,\partial_\chi a'_{01} &= 0\,,\\
\frac{1}{1-\chi}v^{\varepsilon}\,\partial_\chi a'_{10} + \partial_\chi\, a'_{01} - \frac{1}{\chi}u^{\varepsilon}\,\partial_\chi a'_{11} &= 0\,,\\
&\vdots\\
\frac{1}{1-\chi}v^{\varepsilon}\,\partial_\chi a'_{n0} + \partial_\chi\, a'_{n-1\,1} - \frac{1}{\chi}u^{\varepsilon}\,\partial_\chi a'_{n1} &= 0\,,\\
&\vdots\\
\frac{1}{1-\chi}v^{\varepsilon}\,\partial_\chi a'_{k-1\,0} + \partial_\chi\, a'_{k-2\,1} - \frac{1}{\chi}u^{\varepsilon}\,\partial_\chi a'_{k-1\,1} &= 0\,,\\
\frac{1}{1-\chi}v^{\varepsilon}\,\partial_\chi a'_{k0} + \partial_\chi\, a'_{k-1\,1} &= 0\,.
\end{aligned} \tag{6.83}$$

Using the $k=1$ analysis we may immediately solve the first and last of these equations to find that

$$\begin{aligned}
a'_{00} &= u^{\varepsilon}\sum_{\lambda}\big(-b_{1\lambda}+b_{1\,\lambda+1}\big)\tilde{P}^{(\varepsilon)}_{\lambda}\,, \qquad & a'_{01} &= v^{\varepsilon}\sum_{\lambda} b_{1\lambda}\,\tilde{P}^{(\varepsilon)}_{\lambda}\,,\\
a'_{k0} &= \sum_{\lambda}\big(b_{k\,\lambda-1}-b_{k\lambda}\big)\tilde{P}^{(\varepsilon)}_{\lambda}\,, & a'_{k-1\,1} &= v^{\varepsilon}\sum_{\lambda} b_{k\lambda}\,\tilde{P}^{(\varepsilon)}_{\lambda}\,,
\end{aligned} \tag{6.84}$$

for arbitrary $b_{1\lambda}$, $b_{k\lambda}$. Assuming

$$a'_{n-1\,1}(\chi,\chi') = v^{\varepsilon}\sum_{\lambda} b_{n\lambda}\,\tilde{P}^{(\varepsilon)}_{\lambda}(\chi,\chi')\,, \qquad n=1,\dots k\,, \tag{6.85}$$

we may prove by induction that this is consistent with (6.83) for

$$a'_{n0}(\chi,\chi') = \sum_{\lambda}\big(b_{n\,\lambda-1}-b_{n\,\lambda}+u^{\varepsilon}(-b_{n+1\,\lambda}+b_{n+1\,\lambda+1})\big)\tilde{P}^{(\varepsilon)}_{\lambda}(\chi,\chi')\,. \tag{6.86}$$

Substituting (6.85) and (6.86) into (6.82), we may thus find that the full solution to the Ward identities (4.31) is given by

$$G^{(\varepsilon)}_{k}(\chi,\chi';\alpha,\alpha') = \mathcal{F}^{(\varepsilon)}_{k}(\chi,\chi';\alpha,\alpha') + \sum_{n=1}^{k} u^{n\varepsilon}(\alpha\alpha')^{n-1}\mathcal{H}^{(\varepsilon)}_{n}(\chi,\chi';\alpha,\alpha')\,, \tag{6.87}$$

where $\mathcal{F}^{(\varepsilon)}_{k}(\chi,\chi';\alpha,\alpha')$ is given by (6.81) and $\mathcal{H}^{(\varepsilon)}_{j}(\chi,\chi';\alpha,\alpha')$ is given by (6.23) for $b_{1\lambda}\to b_{j\lambda}$. The k different $b_{n\lambda}$ are arbitrary and in four dimensions their contribution corresponds to k different single variable contributions. In fact, for $\varepsilon\neq 1$ we may further use the arbitrariness in $\mathcal{F}_{mn}(\chi,\chi')$ to cancel most $\mathcal{H}^{(\varepsilon)}_{n}$ contributions, similarly to the $k=2$ case.[29] In this way we find (6.1).

[29]This is easily seen as follows: we may redefine $\mathcal{F}_{n-2\,0}$ as in (6.58), for $b_{2\,\lambda}\to b_{n\,\lambda}$, to absorb $\mathcal{H}^{(\varepsilon)}_{n}$, $n\geq 2$, for $\varepsilon\neq 1,2$. After redefining $\mathcal{F}_{n-2\,0}$ in this way, the remaining contributions from $\mathcal{H}^{(\varepsilon)}_{n}$, $n\geq 2$, for $\varepsilon=2$ may be absorbed into $\mathcal{H}^{(\varepsilon)}_{n-1}$ as in (6.61) for $b_{1\,\lambda}\to b_{n-1\,\lambda}$. In this way we end up with only $\mathcal{H}^{(\varepsilon)}_{1}$ contributions - these may be absorbed, as in the $k=2$ case, into $\mathcal{F}_{00}$ for $\varepsilon\neq 1,2$. We stress that for odd dimensions the modification to $\mathcal{F}_{00}$ which cancels $\mathcal{H}^{(\varepsilon)}_{1}$ is defined only in the limit $\mu\to 0$, with μ being some regulating parameter.

We may show directly that the $\mathcal{H}_j(\chi,\chi',\alpha,\alpha')$ contributions to the four-point function of the form

$$G_j(\chi,\chi')\,(\alpha\alpha')^j\,\mathcal{H}_j^{(\varepsilon)}(\chi,\chi';\alpha,\alpha')\,, \tag{6.88}$$

satisfy the Ward identities (4.31), where G_j are some symmetric functions, whereby using

$$\left(\chi\frac{\partial}{\partial\chi}-\varepsilon\,\alpha\frac{\partial}{\partial\alpha}+\varepsilon\right)\mathcal{H}_j^{(\varepsilon)}(\chi,\chi';\alpha,\alpha')\Big|_{\alpha\to 1/\chi}=0\,, \tag{6.89}$$

the Ward identities simply imply that

$$G_j(\chi,\chi')\propto u^{(j+1)\varepsilon}\,. \tag{6.90}$$

The proof of (6.89) is trivial once we show that,

$$\begin{aligned}\left(\chi\frac{\partial}{\partial\chi}-\varepsilon\alpha\frac{\partial}{\partial\alpha}+\varepsilon\right)f_\lambda(\chi,\chi';\alpha)\Big|_{\alpha\to 1/\chi}&=\\ =\frac{1}{2}(\lambda+2\varepsilon)\tilde{P}_\lambda^{(\varepsilon)}(\chi,\chi')+\frac{1}{\chi-\chi'}\frac{1}{2(\lambda+\varepsilon)}&\Big(\lambda(\lambda+1)\tilde{P}_{\lambda+1}^{(\varepsilon)}(\chi,\chi')-\\ &-\chi\chi'(\lambda+2\varepsilon-1)(\lambda+2\varepsilon)\tilde{P}_{\lambda-1}^{(\varepsilon)}(\chi,\chi')\Big)\,,\end{aligned} \tag{6.91}$$

for $f_\lambda(\chi,\chi';\alpha)=\tilde{P}_\lambda^{(\varepsilon)}(\chi,\chi')$ or $f_\lambda(\chi,\chi';\alpha)=\alpha\tilde{P}_{\lambda+1}^{(\varepsilon)}(\chi,\chi')$.

In analogy with the SO(3) general k solution (5.39), we may write (6.81) in a basis independent manner by employing a new operator,

$$\begin{aligned}\mathcal{D}'_\varepsilon=&\left(D_\varepsilon-\varepsilon\frac{1}{v}\left(D_0^+-D_1^++\varepsilon\frac{\partial}{\partial\sigma}\sigma\right)\tau\frac{\partial}{\partial\tau}+\right.\\ &\left.+\varepsilon\frac{1}{uv}\left(-vD_1^++\varepsilon\left(v\frac{\partial}{\partial\sigma}\sigma+\frac{\partial}{\partial\tau}\tau-1\right)\right)\left(\frac{\partial}{\partial\sigma}\sigma+\frac{\partial}{\partial\tau}\tau\right)\right)^{\varepsilon-1},\end{aligned} \tag{6.92}$$

for $\sigma=\alpha\alpha'$, $\tau=(\alpha-1)(\alpha'-1)$, where D_0^+, D_1^+ are given in appendix B, whereby we may write

$$\begin{aligned}\mathcal{F}_k^{(\varepsilon)}(\chi,\chi';\alpha,\alpha')=&\Big(\sigma^2\,\mathcal{D}'_\varepsilon\,uv+\tau^2\,\mathcal{D}'_\varepsilon\,u+\mathcal{D}'_\varepsilon\,v-\sigma\,\mathcal{D}'_\varepsilon\,v(u+1-v)-\\ &-\tau\,\mathcal{D}'_\varepsilon\,(u+v-1)-\sigma\tau\,\mathcal{D}'_\varepsilon\,u(v+1-u)\Big)\mathcal{M}_{k-2}^{(\varepsilon)}(\chi,\chi';\alpha,\alpha')\,,\end{aligned} \tag{6.93}$$

for arbitrary $\mathcal{M}_{k-2}^{(\varepsilon)}(\chi,\chi';\alpha,\alpha')$ of degree $k-2$ in α or α'. For $\varepsilon=1$, when $\mathcal{D}'_\varepsilon\to 1$, then the prefactor in (6.93) factorizes into $(\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)$ and we recover the unrestricted two variable part of the solution in $d=4$, $\mathcal{N}=4$.

7. The conformal partial wave expansion for the $k=2$ case

Due to invariance of the four point function under R symmetry, we may expand

$$G_k^{(\varepsilon)}(\chi,\chi';\alpha,\alpha')=\sum_{a=0}^{k}\sum_{b=0}^{a}Y_{ab}^{(n)}(\alpha,\alpha')\,A_{ab}(\chi,\chi')\,, \tag{7.1}$$

where $Y^{(n)}_{ab}(\alpha,\alpha')$ (of degree at most a in α or α') are eigenfunctions of the SO(n) Casimir operator $L^2 = \frac{1}{2} L^{\mathbf{ij}} L^{\mathbf{ij}}$, where $L^{\mathbf{ij}} = L_1^{\mathbf{ij}} + L_2^{\mathbf{ij}}$ with

$$L^{\mathbf{ij}} = Y^{\mathbf{i}} \frac{\partial}{\partial Y^{\mathbf{j}}} - Y^{\mathbf{j}} \frac{\partial}{\partial Y^{\mathbf{i}}} , \tag{7.2}$$

in terms of the null variables in (2.15), so that

$$L^2 \, Y^{(n)}_{ab}(\alpha,\alpha') = -2\left(a(a+n-3) + b(b+1)\right) Y^{(n)}_{ab}(\alpha,\alpha') . \tag{7.3}$$

An expansion for $Y^{(n)}_{ab}(\alpha,\alpha')$ is given in [22], for general k and n, in terms of Jack polynomials. Other expansions may be found in [15] where these harmonic polynomials are given a more complete treatment.

In terms of superconformal four-point functions, $Y^{(n)}_{ab}(\alpha,\alpha')$ correspond to projections on to irreducible representations in the product of two rank-k symmetric traceless representations of SO(n) whereby for $d=3$ and SO(8) then $(a,b) \to [0, a-b, 2b, 0]$, for $d=4$ and SO(6) then $(a,b) \to [a-b, 2b, a-b]$, for $d=5$ and SO(3) the relevant cases are $(a,a) \to [2a]$,[30] for $d=6$ and SO(5) then $(a,b) \to [2(a-b), 2b]$.

Also, the functions $A_{ab}(\chi,\chi')$ generally admit an expansion in terms of appropriate conformal partial waves $\mathcal{G}^{(\ell)}_{\Delta}(\chi,\chi')$ which then represents a rank-ℓ symmetric traceless operator, of dimension Δ belonging to the (a,b) R symmetry representation, in the operator product expansion of $\mathcal{O}^k(x_1,y_1)\mathcal{O}^k(x_2,y_2)$.

From [22] and in $d = 2(\varepsilon+1)$ dimensions, such conformal partial waves may be expanded as

$$\mathcal{G}^{(\ell)}_{\Delta}(\chi,\chi') = \sum_{m,n\geq 0} r_{nm}(\Delta,\ell)\, P^{(\varepsilon)}_{\frac{1}{2}(\Delta+\ell)+m\, \frac{1}{2}(\Delta-\ell)+n}(\chi,\chi') , \tag{7.4}$$

where the coefficients r_{mn} are given by (we are assuming that the conformal dimensions of the operators at x_1, x_2 are the same, similarly for those at x_3, x_4)

$$r_{mn}(\Delta,\ell) = \left(-\frac{1}{2}\right)^{\ell} \left(\frac{1}{2}(\Delta+\ell)\right)_m^{\,2} \left(\frac{1}{2}(\Delta-\ell)-\varepsilon\right)_n^{\,2} \hat{r}_{mn}(\Delta,\ell) , \tag{7.5}$$

with $\hat{r}_{mn}$ being given recursively by,[31]

$$\begin{aligned}\big(m(m+\Delta+\ell-1) + n(n+\Delta-\ell-2\,\varepsilon-1)\big)\hat{r}_{mn} = {}& \frac{\ell+m-n-1+2\,\varepsilon}{\ell+m-n-1+\varepsilon}\,\hat{r}_{m-1n} + \\ & + \frac{\ell+m-n+1}{\ell+m-n+1+\varepsilon}\,\hat{r}_{mn-1} ,\end{aligned} \tag{7.6}$$

with $\hat{r}_{00} = (2\varepsilon)_\ell/(\varepsilon)_\ell$.

[30] This is related to the fact that for $n=3$ then $\alpha=\alpha'$, and $Y^{(3)}_{ab}(\alpha,\alpha')=0$ unless $b > a-2$. Otherwise, in terms of Legendre polynomials, $Y^{(3)}_{aa}(\alpha,\alpha') \propto P_{2a}(2\,\alpha-1)$, $Y^{(3)}_{a\,a-1}(\alpha,\alpha') \propto P_{2a-1}(2\,\alpha-1)$ [15].

[31] This recurrence relation has been solved explicitly in [22] in terms of ${}_4F_3$ generalized hypergeometric functions.

For the purposes of analysing the $k=2$ case we need only $Y^{(n)}_{ab}(\alpha,\alpha')$, $a=0,1,2$,

$$
\begin{aligned}
&Y^{(n)}_{00}(\alpha,\alpha') = 1\,,\\
&Y^{(n)}_{10}(\alpha,\alpha') = \sigma-\tau\,,\\
&Y^{(n)}_{11}(\alpha,\alpha') = \sigma+\tau-\frac{2}{n}\,,\\
&Y^{(n)}_{20}(\alpha,\alpha') = \sigma^2+\tau^2-2\sigma\,\tau-\frac{2}{n-2}(\sigma+\tau)+\frac{2}{(n-2)(n-1)}\,,\\
&Y^{(n)}_{21}(\alpha,\alpha') = \sigma^2-\tau^2-\frac{4}{n+2}(\sigma-\tau)\,,\\
&Y^{(n)}_{22}(\alpha,\alpha') = \sigma^2+\tau^2+4\sigma\,\tau-\frac{8}{n+4}(\sigma+\tau)+\frac{8}{(n+2)(n+4)}\,,
\end{aligned}
\tag{7.7}
$$

where $\sigma=\alpha\alpha'$ and $\tau=(\alpha-1)(\alpha'-1)$, which have been computed in [15].

In terms of components, for

$$
G^{(\varepsilon)}_2(\chi,\chi';\alpha,\alpha') = \sum_{0\le m+n\le 2}(\alpha\alpha')^n\,\big((\alpha-1)(\alpha'-1)\big)^m\,b_{nm}(u,v)\,,
\tag{7.8}
$$

then

$$
\begin{aligned}
A_{00} &= b_{00}+\frac{1}{n}\,(b_{10}+b_{01})+\frac{2}{(n+2)(n-1)}\,(b_{20}+b_{02})+\frac{n-2}{(n-1)n(n+2)}\,b_{11}\,,\\
A_{11} &= \frac{1}{2}(b_{10}+b_{01})+\frac{2\,n}{(n-2)(n+4)}\,(b_{20}+b_{02})+\frac{n-4}{(n-2)(n+4)}\,b_{11}\,,\\
A_{10} &= \frac{1}{2}(b_{10}-b_{01})+\frac{2}{n+2}\,(b_{20}-b_{02})\,,\\
A_{20} &= \frac{1}{3}(b_{20}+b_{02})-\frac{1}{6}\,b_{11}\,,\\
A_{21} &= \frac{1}{2}(b_{20}-b_{02})\,,\\
A_{22} &= \frac{1}{6}(b_{20}+b_{11}+b_{02})\,,
\end{aligned}
\tag{7.9}
$$

so that we may easily read off various contributions from (6.64).

7.1 The analysis of the unrestricted two-variable part

We start with contributions of long superconformal multiplets, where the dimensions of operators contributing in the operator product expansion are unrestricted, save for unitarity constraints. The corresponding contributions from the two-variable functions to the projections on to the various R symmetry channels are given by, from (6.57), with (7.9),

$$
\begin{aligned}
A^{\rm long}_{22} &= \frac{1}{6}\,u^{2\varepsilon}\,\Delta_\varepsilon\,u^2\,a\,,\\
A^{\rm long}_{21} &= \frac{1}{2}\,u^{2\varepsilon}\,\Delta_\varepsilon\,u(v-1)\,a\,,\\
A^{\rm long}_{20} &= \frac{1}{6}\,u^{2\varepsilon}\,\Delta_\varepsilon\,u(3(v+1)-u)\,a\,,
\end{aligned}
$$

$$
\begin{aligned}
A_{10}^{\text{long}} &= \frac{1}{2}\, u^{2\varepsilon}\, \Delta_\varepsilon\, (v-1) \left((v+1) - \frac{n-2}{n+2}\, u \right) a \,, \\
A_{11}^{\text{long}} &= \frac{1}{2}\, u^{2\varepsilon}\, \Delta_\varepsilon \left((v-1)^2 - \frac{n-4}{n-2}\, u\,(v+1) + \frac{2(n-4)}{(n-2)(n+4)}\, u^2 \right) a \,, \\
A_{00}^{\text{long}} &= \frac{1}{4}\, u^{2\varepsilon}\, \Delta_\varepsilon \Bigg((v+1)^2 - \frac{n-4}{n} (v-1)^2 - \\
&\qquad\qquad - \frac{4(n-2)}{n(n-1)}\, u\,(v+1) + \frac{4(n-2)}{n(n-1)(n+2)}\, u^2 \Bigg) a \,. \qquad (7.10)
\end{aligned}
$$

The simplest possibility is to expand

$$
u^2\, a = (\Delta_\varepsilon)^{-1} \sum_{m,n \geq 0} r_{mn}(\Delta,\ell)\, P^{(\varepsilon)}_{\alpha_+ + m\, \alpha_- + n}(\chi,\chi')\,, \qquad \alpha_\pm = \frac{1}{2}(\Delta \pm \ell) - 2\,\varepsilon\,, \qquad (7.11)
$$

so that

$$
A_{22}^{\text{long}}(u,v) = \frac{1}{6}\, \mathcal{G}_\Delta^{(\ell)}(\chi,\chi')\,, \qquad (7.12)
$$

representing just a single operator in the $(2,2)$ R symmetry channel. Using results derived in appendix D we may find that, with the choice of a as in (7.11),

$$
\begin{aligned}
A_{21}^{\text{long}} = {} & \frac{\Delta-\ell-4\varepsilon}{\Delta-\ell-2\varepsilon-2}\, \mathcal{G}_{\Delta-1}^{(\ell+1)} + \frac{1}{4}\, \frac{\Delta+\ell-2\varepsilon}{\Delta+\ell-2}\, \frac{\ell(\ell+2\varepsilon-1)}{(\ell+\varepsilon-1)(\ell+\varepsilon)}\, \mathcal{G}_{\Delta-1}^{(\ell-1)} + \\
& + \frac{1}{16}\, \frac{(\Delta+\ell)(\Delta+\ell+2\varepsilon-2)}{(\Delta+\ell-1)(\Delta+\ell+1)}\, \frac{(\Delta-1)(\Delta-2\varepsilon)}{(\Delta-\varepsilon-1)(\Delta-\varepsilon)}\, \mathcal{G}_{\Delta+1}^{(\ell+1)} + \\
& + \frac{1}{64}\, \frac{(\Delta-\ell-2)(\Delta-\ell-2\varepsilon)}{(\Delta-\ell-2\varepsilon+1)(\Delta-\ell-2\varepsilon-1)}\, \frac{\ell(\ell+2\varepsilon-1)}{(\ell+\varepsilon-1)(\ell+\varepsilon)} \times \\
& \times \frac{(\Delta-1)(\Delta-2\varepsilon)}{(\Delta-\varepsilon-1)(\Delta-\varepsilon)}\, \mathcal{G}_{\Delta+1}^{(\ell-1)}\,, \qquad (7.13)
\end{aligned}
$$

and this satisfies positivity for the twist $\Delta - \ell \geq 4\varepsilon$. Other expansions may be found in appendix D.

The spectrum of operators appearing in the conformal partial wave expansion of A_R^{long} is consistent with that of a long superconformal-multiplet with quasi-primary field in the $(0,0)$ (singlet) representation of $\mathrm{SO}(n)$. From the conformal partial wave expansion, the spectrum expected is, for $(\Delta')_{\ell'}$ representing an operator of dimension Δ' and in the spin representation labelled by ℓ',

R	$(0,0)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,1)$	$(2,2)$
	$(\Delta)_{\ell,\ell\pm 2,\ell\pm 4}$	$(\Delta\pm 1)_{\ell\pm 1,\ell\pm 3}$	$(\Delta)_{\ell,\ell\pm 2}$	$(\Delta)_{\ell,\ell\pm 2}$	$(\Delta\pm 1)_{\ell\pm 1}$	$(\Delta)_\ell$
	$(\Delta\pm 2)_{\ell,\ell\pm 2}$	$(\Delta\pm 3)_{\ell\pm 1}$	$(\Delta\pm 2)_{\ell\pm 2,\ell}$	$(\Delta\pm 2)_\ell$		
	$(\Delta\pm 4)_\ell$					

which is consistent with what we expect from the OPE [35] as shown from the relevant representation theory in E. That the spectrum of operators (7.14) arises from the conformal partial wave expansion of (7.10) is essentially shown in appendix D. These results are also consistent with the corresponding case for $d = 4, \mathcal{N} = 4$ considered in [11].

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7.2 The analysis of $\mathcal{H}_1^{(\varepsilon)}$ for general ε

Much simpler to analyse are the contributions to the conformal partial wave expansion coming from the $\mathcal{H}_1$ parts of (6.57) which turn out to correspond to twist 2ε operators. With the definition (7.4), we may write such conformal partial waves simply as,

$$\mathcal{G}^{(\ell)}_{\ell+2\varepsilon}(\chi,\chi') = u^\varepsilon \sum_{m=0}^{\infty} r_m(\ell)\, \tilde{P}^{(\varepsilon)}_{\ell+m}(\chi,\chi')\,, \tag{7.14}$$

where,

$$r_m(\ell) = \Gamma(\varepsilon)\left(-\frac{1}{2}\right)^{\ell} \frac{(\ell+\varepsilon)_m(\ell+\varepsilon)_{m+1-\varepsilon}}{m!(2\ell+2\varepsilon)_m}\,. \tag{7.15}$$

With the choice,

$$b_{1\,\lambda} = 2\sum_{m=0}^{\infty} r_m(\ell+2)\,\delta_{\lambda,\ell+m+1}\,, \tag{7.16}$$

and using the following two relations, namely,

$$\begin{aligned} u^\varepsilon \sum_{m=0}^{\infty}\big(r_m(\ell) - 2\,r_{m+1}(\ell)\big)\tilde{P}^{(\varepsilon)}_{\ell+m}(\chi,\chi') \;=\; & \frac{\ell}{\ell+\varepsilon-1}\,\mathcal{G}^{(\ell-1)}_{\ell+2\varepsilon-1} + \\ & + \frac{(\ell+\varepsilon)(\ell+2\varepsilon-1)}{(2\ell+2\varepsilon-1)(2\ell+2\varepsilon+1)}\,\mathcal{G}^{(\ell+1)}_{\ell+2\varepsilon+1}\,, \end{aligned} \tag{7.17}$$

and

$$\begin{aligned} u^\varepsilon \sum_{m=0}^{\infty}\big(r_{m+2}(\ell) - r_{m+1}(\ell)\big)\tilde{P}^{(\varepsilon)}_{\ell+m}(\chi,\chi') \;=\; & \frac{\ell(\ell-1)}{4(\ell+\varepsilon-2)(\ell+\varepsilon-1)}\mathcal{G}^{(\ell-2)}_{\ell+2\varepsilon-2} - \\ & - \frac{(\ell+\varepsilon)(\ell+\varepsilon-1)+\varepsilon(\varepsilon-2)}{2(2\ell+2\varepsilon-3)(2\ell+2\varepsilon+1)}\mathcal{G}^{(\ell)}_{\ell+2\varepsilon} + \\ & + \frac{(\ell+\varepsilon)(\ell+\varepsilon+1)(\ell+2\varepsilon-1)(\ell+2\varepsilon)}{4(2\ell+2\varepsilon-1)(2\ell+2\varepsilon+1)^2(2\ell+2\varepsilon+3)}\mathcal{G}^{(\ell+2)}_{\ell+2\varepsilon+2}, \end{aligned} \tag{7.18}$$

we may obtain the following contributions to $A_R(\chi,\chi')$,

$$\begin{aligned} & A_{11} \sim \mathcal{G}^{(\ell+2)}_{\ell+2\varepsilon+2}\,, \qquad A_{10} \sim \frac{\ell+2}{\ell+\varepsilon+1}\,\mathcal{G}^{(\ell+1)}_{\ell+2\varepsilon+1} + \frac{(\ell+\varepsilon+2)(\ell+2\varepsilon+1)}{(2\ell+2\varepsilon+3)(2\ell+2\varepsilon+5)}\,\mathcal{G}^{(\ell+3)}_{\ell+2\varepsilon+3}\,, \\ & A_{00} \sim \frac{(\ell+2)(\ell+1)}{2(\ell+\varepsilon)(\ell+\varepsilon+1)}\mathcal{G}^{(\ell)}_{\ell+2\varepsilon} + \left(\frac{2}{n} - \frac{(\ell+\varepsilon+2)(\ell+\varepsilon+1)+\varepsilon(\varepsilon-2)}{(2\ell+2\varepsilon+1)(2\ell+2\varepsilon+5)}\right)\mathcal{G}^{(\ell+2)}_{\ell+2\varepsilon+2} + \\ & \qquad + \frac{(\ell+\varepsilon+2)(\ell+\varepsilon+3)(\ell+2\varepsilon+1)(\ell+2\varepsilon+2)}{2(2\ell+2\varepsilon+3)(2\ell+2\varepsilon+5)^2(2\ell+2\varepsilon+7)}\mathcal{G}^{(\ell+4)}_{\ell+2\varepsilon+4}\,, \end{aligned} \tag{7.19}$$

with other contributions vanishing. For $(\varepsilon,n) = (1,6)$ this agrees with similar results in [11] where the relevant contribution was denoted by $\mathcal{B}_\ell$ and the corresponding operators belong to a semi-short $\mathcal{N}=4$ superconformal multiplet whereby the quasi-primary field has $(\Delta',\ell') = (\ell+2,\ell)$ and is an $\mathrm{SU}(4)_R$ singlet. The coefficients in (7.19) are positive for the range of (ε,n) values of relevance and the lowest dimension operator contributing is again an R symmetry singlet with $(\Delta',\ell') = (\ell+2\varepsilon,\ell)$. For $(\varepsilon,n) = (\frac{1}{2},8)$ the contribution from the singlet operator with $(\Delta',\ell') = (\ell+3,\ell+2)$ is absent.

7.3 Modifications for $d = 4$ and $\mathcal{N} = 4$

For $(\varepsilon, n) = (1,6), (2,5)$ making the modification (6.58) does not cancel all the $\mathcal{H}_2$ contributions in (6.57). We have already dealt with those for $(\varepsilon, n) = (2,5)$. For $(\varepsilon, n) = (1,6)$, those left over may be easily computed to be given by

$$G_2^{(1)}(\chi,\chi';\alpha,\alpha') \sim \sum_\lambda b_{2\,\lambda}\left((\alpha+\alpha')\,u\,P^{(1)}_{\lambda+1}(\chi,\chi';\alpha,\alpha') - P^{(1)}_{\lambda+2}(\chi,\chi';\alpha,\alpha')\right). \tag{7.20}$$

We may make contact with known results [11] by re-defining $b_{1\,\lambda}$ so that,

$$b_{1\,\lambda} \to b_{1\,\lambda} - b_{2\,\lambda-1}, \tag{7.21}$$

and choosing

$$b_{2\,\lambda} = 2\sum_{m=0}^{\infty} r_m(\ell+1)\,\delta_{\lambda,\ell+m-1}\,. \tag{7.22}$$

We may then find that the conformal partial wave expansion of the remaining $b_{2\,\lambda}$ contributions is given by,

$$\begin{aligned}
A_{21} &\sim \mathcal{G}^{(\ell+1)}_{\ell+3}\,, \qquad A_{20} \sim \mathcal{G}^{(\ell)}_{\ell+2} + \frac{(\ell+2)^2}{(2\ell+3)(2\ell+5)}\,\mathcal{G}^{(\ell+2)}_{\ell+4}\,,\\
A_{11} &\sim 2\,\mathcal{G}^{(\ell+2)}_{\ell+2} + \frac{1}{2}\,\mathcal{G}^{(\ell)}_{\ell+2} + \frac{(\ell+2)^2}{2(2\ell+3)(2\ell+5)}\,\mathcal{G}^{(\ell+2)}_{\ell+4}\,,\\
A_{10} &\sim 2\,\mathcal{G}^{(\ell+1)}_{\ell+1} + \frac{2(\ell+2)^2}{(2\ell+3)(2\ell+5)}\,\mathcal{G}^{(\ell+3)}_{\ell+3} + \frac{1}{2}\,\mathcal{G}^{(\ell-1)}_{\ell+1} + \\
&\quad + \frac{(2\ell+3)^2-3}{4(2\ell+1)(2\ell+5)}\,\mathcal{G}^{(\ell+1)}_{\ell+3} + \frac{(\ell+2)^2(\ell+3)^2}{2(2\ell+3)(2\ell+5)^2(2\ell+7)}\,\mathcal{G}^{(\ell+3)}_{\ell+5}\\
A_{00} &\sim \mathcal{G}^{(\ell)}_{\ell} + \frac{2(\ell+1)(\ell+2)}{3(2\ell+1)(2\ell+5)}\,\mathcal{G}^{(\ell+2)}_{\ell+2} + \frac{(\ell+2)^2(\ell+3)^2}{(2\ell+3)(2\ell+5)^2(2\ell+7)}\,\mathcal{G}^{(\ell+4)}_{\ell+4} + \\
&\quad + \frac{1}{15}\,\mathcal{G}^{(\ell)}_{\ell+2} + \frac{(\ell+2)^2}{15(2\ell+3)(2\ell+5)}\,\mathcal{G}^{(\ell+2)}_{\ell+4}\,,
\end{aligned} \tag{7.23}$$

so that we have only twist zero and twist one contributions. To achieve this form we have used (7.17), (7.18) and

$$\sum_{m=0}^{\infty} r_{m-1}(\ell)\,\tilde{P}^{(1)}_{\ell+m}(\chi,\chi') = \frac{1}{2}\,\mathcal{G}^{(\ell)}_{\ell+2}(\chi,\chi') - 2\,\mathcal{G}^{(\ell+1)}_{\ell+1}(\chi,\chi')\,, \tag{7.24}$$

and the following trivial identities, namely,

$$r_{m-1}(\ell) - 2r_m(\ell) = r_m(\ell-1) + \frac{(\ell+1)^2}{(2\ell+1)(2\ell+3)}\,r_{m-2}(\ell+1)\,, \tag{7.25}$$

(which, when combined with (7.24), is useful for determining A_{10}) and

$$r_{m+1}(\ell) - r_m(\ell) = \frac{1}{4}\,r_{m+1}(\ell-2) + \frac{\ell(\ell+1)-1}{6(2\ell-1)(2\ell+3)}\,r_{m-1}(\ell)$$

$$+\frac{(\ell+1)^2(\ell+2)^2}{4(2\ell+1)(2\ell+3)^2(2\ell+5)}\, r_{m-3}(\ell+2)\,, \tag{7.26}$$

(which, again when combined with (7.24), is useful for determining A_{00}).

The expansion (7.23) is significant for four dimensions because it in fact represents contributions from a certain $\mathcal{N}=4$ non-unitary multiplet, starting from an $\mathrm{SU}(4)_R$ singlet. This point and the construction of such multiplets has been discussed in [36]. It is a general feature of the $d=4$ $\mathcal{N}=4$ superconformal four-point functions discussed here, for the single variable parts of the Ward identity solutions [15].

Acknowledgments

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A. Superconformal invariants for the case $d=4$ $\mathcal{N}=4$

In the main text of this paper we needed just the linearized superconformal invariants $\hat{z}, \hat{z}', \hat{w}, \hat{w}'$. They could be easily obtained by examining the linearized supersymmetry transformations. In principle, the same procedure could give the full non-linear version of these invariants. However, there exists an alternative and more efficient method which we are going to apply to the case $d=4$ $\mathcal{N}=4$.[32] In [12] it was pointed out that $\hat{z}, \hat{z}', \hat{w}, \hat{w}'$ can be obtained as the eigenvalues of the four-point covariant

$$X_{12}X_{23}^{-1}X_{34}X_{41}^{-1}\,, \tag{A.1}$$

where the supermatrices

$$X^{AB'} = \begin{pmatrix} x^{\alpha\beta'} & \theta^{\alpha b'} \\ \bar{\theta}^{a\beta'} & y^{ab'} \end{pmatrix}, \tag{A.2}$$

combine the even and odd variables at each point of analytic superspace.

Here we are going to this explicitly, with two additional important simplifications. Firstly, we use the superconformal transformations to fix the superspace frame discussed in the main text:

$$X_2 = \Sigma\,, \quad X_3 = X_4^{-1} = 0\,. \tag{A.3}$$

Here Σ is a constant supermatrix obtained by setting $\theta_2 = 0$ and $x_2^\mu = (1, 0, \ldots, 0)$, $y_2^i = (1, 0, \ldots, 0)$. Then the covariant (A.1) is reduced to just $X \equiv X_{12}$. Secondly, instead of the explicit diagonalization of this matrix proposed in [12], it is easier to solve the eigenvalue equation

$$\mathrm{Ber}(X - \lambda\mathbb{I}) = 0\,, \tag{A.4}$$

[32] E.S. acknowledges an enlightening discussion on this point with Paul Howe.

where Ber is the Berezinian (superdeterminant) and λ are the eigenvalues. To zeroeth order in the θ expansion these eigenvalues coincide with the even variables z, z', w, w'. Their superconformal completions are then obtained as "perturbations" around z, z', w, w'.

A.1 The eigenvalues of the supermatrix

For a general supermatrix X of $\mathrm{GL}(n|m)$ given in block notation

$$X = \begin{pmatrix} Z & \Theta \\ \bar{\Theta} & W \end{pmatrix}, \tag{A.5}$$

the Berezinian (superdeterminant) is defined as

$$\mathrm{Ber}(X) = \frac{\mathrm{Det}(Z - \Theta W^{-1}\bar{\Theta})}{\mathrm{Det}(W)} = \frac{\mathrm{Det}(Z)}{\mathrm{Det}(W - \bar{\Theta} Z^{-1}\Theta)}, \tag{A.6}$$

The Berezinian is a multiplicative function, invariant under conjugation, e.g.,

$$\mathrm{Ber}(XY) = \mathrm{Ber}(X)\mathrm{Ber}(Y), \qquad \mathrm{Ber}(MXM^{-1}) = \mathrm{Ber}(X). \tag{A.7}$$

As a consequence, the roots and poles of the rational function $\mathrm{Ber}(X - \lambda\mathbb{I})$ are invariant under conjugation. The eigenvalue equation for the supermatrix (A.2) reads

$$\begin{aligned} 0 &= \mathrm{Ber}(X_{12} - \lambda\mathbb{I}) \\ &= \left[\left((z-\lambda) - \frac{\theta^{11}\bar{\theta}^{11}}{w-\lambda} - \frac{\theta^{12}\bar{\theta}^{21}}{w'-\lambda}\right)\left((z'-\lambda) - \frac{\theta^{21}\bar{\theta}^{12}}{w-\lambda} - \frac{\theta^{22}\bar{\theta}^{22}}{w'-\lambda}\right) - \right. \\ &\quad \left. - \left(\frac{\theta^{11}\bar{\theta}^{12}}{w-\lambda} + \frac{\theta^{12}\bar{\theta}^{22}}{w'-\lambda}\right)\left(\frac{\theta^{21}\bar{\theta}^{11}}{w-\lambda} + \frac{\theta^{22}\bar{\theta}^{21}}{w'-\lambda}\right)\right] / (w-\lambda)(w'-\lambda). \end{aligned} \tag{A.8}$$

Thus, the invariants (eigenvalues) are the two roots and two poles of the rational fraction above. The first eigenvalue is found as an expansion around z

$$\hat{z} = z - \Delta z = z - \Delta z_1 - \Delta z_2 - \Delta z_3 - \Delta z_4. \tag{A.9}$$

It is a root of the Berezinian or equivalently of the polynomial of degree six given by the numerator of the Berezinian:

$$\begin{aligned} &\Big((z-\lambda)(w-\lambda)\Big(w'-\lambda\Big) - \theta^{11}\bar{\theta}^{11}(w'-\lambda) - \theta^{12}\bar{\theta}^{21}(w-\lambda)\Big) \times \\ &\times \left((z'-\lambda)(w-\lambda)(w'-\lambda) - \theta^{21}\bar{\theta}^{12}(w'-\lambda) - \theta^{22}\bar{\theta}^{22}(w-\lambda)\right) - \\ &- \left(\theta^{11}\bar{\theta}^{12}(w'-\lambda) + \theta^{12}\bar{\theta}^{22}(w-\lambda)\right)\left(\theta^{21}\bar{\theta}^{11}(w'-\lambda) + \theta^{22}\bar{\theta}^{21}(w-\lambda)\right) = 0 \end{aligned} \tag{A.10}$$

This equation may be written in a more suggestive form using the variable Δz

$$\begin{aligned} &\left(\Delta z\left(1 + \frac{\Delta z}{w-z}\right)\left(1 + \frac{\Delta z}{w'-z}\right) - \frac{\theta^{11}\bar{\theta}^{11}}{w-z} - \frac{\theta^{12}\bar{\theta}^{21}}{w'-z}\right) \times \\ \times &\left(\left(1 + \frac{\Delta z}{z'-z}\right)\left(1 + \frac{\Delta z}{w-z}\right)\left(1 + \frac{\Delta z}{w'-z}\right) - \frac{\theta^{21}\bar{\theta}^{12}}{(z'-z)(w-z)} - \frac{\theta^{22}\bar{\theta}^{22}}{(z'-z)(w'-z)}\right) - \end{aligned}$$

$$-\left(\frac{\theta^{11}\bar\theta^{12}}{(w-z)(z'-z)^{\frac12}}\left(1+\frac{\Delta z}{w'-z}\right)+\frac{\theta^{12}\bar\theta^{22}}{(w'-z)(z'-z)^{\frac12}}\left(1+\frac{\Delta z}{w-z}\right)\right)\times \tag{A.11}$$
$$\times\left(\frac{\theta^{21}\bar\theta^{11}}{(w-z)(z'-z)^{\frac12}}\left(1+\frac{\Delta z}{w'-z}\right)+\frac{\theta^{22}\bar\theta^{21}}{(w'-z)(z'-z)^{\frac12}}\left(1+\frac{\Delta z}{w-z}\right)\right)=0$$

This form makes explicit the fact that each θ will appear in Δz dressed with some factor. As an example, θ^{11} and $\bar\theta^{11}$ are associated with the factor $(w-z)^{-1/2}$ while θ^{22} and $\bar\theta^{22}$ are associated with $((w'-z)(z'-z))^{-1/2}$. The invariant $\hat z$ is then found iteratively as an expansion in the θs and the net result is

$$\begin{aligned}
\Delta z_1 &= \frac{\theta^{11}\bar\theta^{11}}{w-z}+\frac{\theta^{12}\bar\theta^{21}}{w'-z}\,,\\
\Delta z_2 &= -\left(\frac{1}{w-z}+\frac{1}{w'-z}\right)\frac{\theta^{11}\bar\theta^{11}\theta^{12}\bar\theta^{21}}{(w-z)(w'-z)}-\\
&\quad-\frac{1}{z'-z}\left(\frac{\theta^{11}\bar\theta^{11}\theta^{21}\bar\theta^{12}}{(w-z)^2}+\frac{\theta^{22}\bar\theta^{22}\theta^{12}\bar\theta^{21}}{(w'-z)^2}-\frac{\theta^{11}\bar\theta^{12}\theta^{22}\bar\theta^{21}}{(w-z)(w'-z)}-\frac{\theta^{12}\bar\theta^{22}\theta^{21}\bar\theta^{11}}{(w-z)(w'-z)}\right),\\
\Delta z_3 &= -\frac{\theta^{22}\bar\theta^{22}\theta^{21}\bar\theta^{12}}{(z'-z)^2(w-z)(w'-z)}\left(\frac{\theta^{11}\bar\theta^{11}}{w-z}+\frac{\theta^{12}\bar\theta^{21}}{w'-z}\right)+\\
&\quad+\frac{\theta^{11}\bar\theta^{11}\theta^{12}\bar\theta^{21}}{(w-z)(w'-z)}\left[\left(\frac{1}{z'-z}+\frac{2}{w-z}+\frac{1}{w'-z}\right)\frac{\theta^{21}\bar\theta^{12}}{(z'-z)(w-z)}+\right.\\
&\qquad\left.+\left(\frac{1}{z'-z}+\frac{1}{w-z}+\frac{2}{w'-z}\right)\frac{\theta^{22}\bar\theta^{22}}{(z'-z)(w'-z)}\right],\\
\Delta z_4 &= 0\,.
\end{aligned}\tag{A.12}$$

Notice that the expansion is truncated and the top term $(\theta)^8$ is absent. We shall come back to this result later on, when discussing the discrete symmetries of the problem.

A.2 Discrete symmetries

Conjugation of the space-time coordinates: this is generated by conjugation with the matrix $\mathcal{M}_S$;

$$\mathcal{M}_S=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\qquad X_S=\mathcal{M}_S X\mathcal{M}_S^{-1}=\begin{pmatrix}z'&0&\theta^{21}&\theta^{22}\\0&z&\theta^{11}&\theta^{12}\\\bar\theta^{12}&\bar\theta^{11}&w&0\\\bar\theta^{22}&\bar\theta^{21}&0&w'\end{pmatrix},\tag{A.13}$$

This is a symmetry due to the identity

$$\mathrm{Ber}(X_S-\lambda\mathbb{I})=\mathrm{Ber}(X-\lambda\mathbb{I})\,.\tag{A.14}$$

Conjugation of the internal coordinates: this is generated by conjugation with the matrix $\mathcal{M}_H$;

$$\mathcal{M}_H=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\qquad X_H=\mathcal{M}_H X\mathcal{M}_H^{-1}=\begin{pmatrix}z&0&\theta^{12}&\theta^{11}\\0&z'&\theta^{22}&\theta^{21}\\\bar\theta^{21}&\bar\theta^{22}&w'&0\\\bar\theta^{11}&\bar\theta^{12}&0&w\end{pmatrix}.\tag{A.15}$$

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This is a symmetry due to the identity

$$\mathrm{Ber}(X_H - \lambda\mathbb{I}) = \mathrm{Ber}(X - \lambda\mathbb{I})\,. \tag{A.16}$$

Exchange of internal and space-time cross ratios: this symmetry is specific to the $\mathcal{N}=4$ case. It is generated, externally to $gl(2|2)$, by conjugation with the matrix $\mathcal{J}$;

$$\mathcal{J} = \begin{pmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{pmatrix}, \qquad X_J = \mathcal{J} X \mathcal{J}^{-1} = \begin{pmatrix} w & 0 & \bar\theta^{11} & \bar\theta^{12} \\ 0 & w' & \bar\theta^{21} & \bar\theta^{22} \\ \theta^{11} & \theta^{12} & z & 0 \\ \theta^{21} & \theta^{22} & 0 & z' \end{pmatrix}, \tag{A.17}$$

or, using 2×2 block notation,

$$X_J = \begin{pmatrix} W & \bar\Theta \\ \Theta & Z \end{pmatrix}. \tag{A.18}$$

From the definition of the Berezinian one obtains

$$\mathrm{Ber}(X_J - \lambda\mathbb{I}) = \mathrm{Ber}^{-1}(X - \lambda\mathbb{I})\,, \tag{A.19}$$

so that this is also a discrete symmetry acting on the superconformal invariants. For example, one has $J(z) = w$, $J(\hat z) = \hat w$, $J(\theta^{ij}) = \bar\theta^{ij}$ and so on.

How these symmetries are used: these symmetries can be used to obtain $\hat z'$, $\hat w$ and $\hat w'$ from $\hat z$. As an example, using the expression for Δz_1 and the conjugation for the space-time variables gives

$$\Delta z_1' = \frac{\theta^{21}\bar\theta^{12}}{w - z'} + \frac{\theta^{22}\bar\theta^{22}}{w' - z'}\,. \tag{A.20}$$

The exchange symmetry may be used, in particular, to understand why the last term Δz_4 in (A.12) vanishes while the corresponding top term is non-vanishing in the $\mathcal{N}=2$ case (see (4.13)). The most general form of the top term is

$$\Delta z_4 = \theta^{[8]} f_z(z, z', w, w')\,, \tag{A.21}$$

where $\theta^{[8]} = \theta^{11}\bar\theta^{11}\theta^{12}\bar\theta^{21}\theta^{21}\bar\theta^{12}\theta^{22}\bar\theta^{22}$. Note that $\theta^{[8]}$ is invariant under the discrete symmetries S, H, and J.

From equation (A.11) we know that the Δz_i depend only on $z'-z$, $w-z$ and $w'-z$, and analogously for $\Delta z_4'$, etc., so that one has

$$\begin{aligned} \Delta z_4 &= \theta^{[8]} F_z(z'-z, w-z, w'-z)\,, & \Delta z_4' &= \theta^{[8]} F_z'(z-z', w-z', w'-z')\,, \\ \Delta w_4 &= \theta^{[8]} F_w(w'-w, z-w, z'-w)\,, & \Delta w_4' &= \theta^{[8]} F_{w'}(w-w', z-w', z'-w')\,. \end{aligned} \tag{A.22}$$

Let us now impose the discrete symmetries. First, the conjugation of the internal coordinates H leaves z and z' invariant while exchanging w and w', so we deduce that both functions F_z and F_z' are symmetric under the exchange of the last two arguments. The

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same result for F_w and $F_{w'}$ is obtained by the use of the conjugation of spatial coordinates S. The conjugations S and H also imply the following identities between functions

$$F_z = F_z' \,, \qquad F_w = F_{w'} \,, \tag{A.23}$$

respectively, so that we are left with only two functions.

The difference between the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ cases is that in the latter case these two functions have to be equal because of the exchange symmetry J. Since this symmetry exchanges spatial and internal coordinates, we have

$$\mathcal{F} = F_z = F_z' = F_w = F_{w'} \,, \tag{A.24}$$

so that we are left with only one function $\mathcal{F}$. The function $\mathcal{F}$ has to satisfy various constraints:

(A) that the supertrace of the matrix X has a vanishing $\theta^{[8]}$ term so that $\Delta z_4 + \Delta z_4' - \Delta w_4 - \Delta w_4' = 0$, hence the following linear equation is satisfied

$$\begin{aligned} &\mathcal{F}(z'-z, w-z, w'-z) + \mathcal{F}(z-z', w-z', w'-z') = \\ &\mathcal{F}(w'-w, z-w, z'-w) + \mathcal{F}(w-w', z-w', z'-w') \,; \end{aligned} \tag{A.25}$$

(B) (as already mentioned) that the function $\mathcal{F}$ is symmetric under the exchange of its last two arguments, $\mathcal{F}(a,b,c) = \mathcal{F}(a,c,b)$;

(C) from the perturbative approach, that the function $\mathcal{F}$ is a rational function with monomials of global degree seven, in order for Δz_4 to have the same dimension as z.

Moreover, we have already seen that with each θ is associated a factor such that the rational factor associated with $\theta^{[8]}$ is $(z'-z)^{-2}(w-z)^{-2}(w'-z)^{-2}$. Hence, we deduce from this result and previous requirements that the function $\mathcal{F}$ must be of the type

$$\mathcal{F}(z'-z, w-z, w'-z) = \frac{1}{(z-z')^2(w-z)^2(w'-z)^2}\Big(- a\frac{1}{z-z'} + b\Big(\frac{1}{w-z} + \frac{1}{w'-z}\Big)\Big) \,, \tag{A.26}$$

where a and b are complex parameters.

A glance at eq. (A.25) shows that such a function does not satisfy the constraint coming from the exchange symmetry unless it vanishes.

A.3 Singularities

The variables u and v. The singularity in $z'-z$ should not be present in physical quantities. In particular, it should be absent from the supersymmetrized cross-ratios $\hat{u}$ and $\hat{v}$. Indeed, direct inspection shows that the $\hat{u}$ and $\hat{v}$ do not have this singularity. A simple argument explains this fact. It makes use of the properties of the matrix X and of the symmetry by exchange of space-time and internal variables. Recall that

$$\hat{u} = \frac{\hat{z}\hat{z}'}{(1+\hat{z})(1+\hat{z}')} \,, \qquad \hat{v} = \frac{1}{(1+\hat{z})(1+\hat{z}')} \,, \tag{A.27}$$

so that one has to check that the singularity is absent from $\hat{z} + \hat{z}'$ and $\hat{z}\hat{z}'$. These two combinations are related to the supertrace and the Berezinian of the matrix X, respectively:

$$\begin{aligned} \hat{z} + \hat{z}' &= \hat{w} + \hat{w'} + \mathrm{Str}(X) \\ \hat{z}\hat{z}' &= \hat{w}\hat{w'}\,\mathrm{Ber}(X)\,. \end{aligned} \tag{A.28}$$

Now, from the expression of $\hat{z}$ and the use of the various discrete symmetries, one gets that, apart from singularities of space-internal type (e.g., $w - z$, etc.), $\hat{z}$ and $\hat{z}'$ have only the space-space singularity $z' - z$ while $\hat{w}$ and $\hat{w'}$ have only the internal-internal singularity $w' - w$. The supertrace and Berezinian of the matrix X are free from these two types of singularities. Hence, eqs. (A.28) tell us that the quantities $\hat{z} + \hat{z}'$, $\hat{z}\hat{z}'$, $\hat{w} + \hat{w'}$ and $\hat{w}\hat{w'}$ have neither $z' - z$, nor $w' - w$ singularity but only singularities of the spatial-internal type.

General functions of $\hat{z}$ and $\hat{z}'$. Let F be a general function of $\hat{z}$ and $\hat{z}'$. In principle, this function will have some singularity in $z' - z$ unless it satisfies some requirements to be derived below. The quantity $F(\hat{z}, \hat{z}')$ is an expansion in θs

$$F(\hat{z}, \hat{z}') = F(z, z') + F^{[2]} + F^{[4]} + F^{[6]} + F^{[8]} \tag{A.29}$$

which can be obtained as a Taylor expansion around $F^{[0]} = F(z, z')$. Due to its length, we shall not write the expansion explicitly but just the conditions for the absence of singularities that we have found.

The term with two θs

$$F^{[2]} = -\left(\Delta z_1 F_z + \Delta z_1' F_{z'}\right) \tag{A.30}$$

is non-singular in $z' - z$ because Δz_1 and $\Delta z_1'$ are not. The condition coming from the four-θ term is

$$\left.\frac{F_{z'} - F_z}{z' - z}\right|_{z'-z\to 0} = \left.\frac{2F_{z'-z}}{z' - z}\right|_{z'-z\to 0}, \tag{A.31}$$

is non-singular. In the case $\mathcal{N} = 2$ the expansion terminates at this level. The $\mathcal{N} = 4$ case is more complicated since higher-order singularities may be present in the six- and eight-θ terms. In the six-θ terms, a $(z' - z)^{-2}$ singularity could be present but is actually removed by the preceding requirement. The $(z' - z)^{-1}$ singularity is removed by requiring that

$$\left.\partial_{z'-z}\left(\frac{F_{z'-z}}{z' - z}\right)\right|_{z'-z\to 0} \tag{A.32}$$

must be non-singular. In the eight-θ term, the possible $(z' - z)^{-2}$ singularity is just not present and the $(z' - z)^{-1}$ singularity is removed by requiring that

$$\left.\frac{(F_{z'} - F_z)_{zz'}}{z' - z}\right|_{z'-z\to 0} \tag{A.33}$$

be non-singular. This is true if no $(z' - z)^3$ term is present in the expansion of F, which holds if F is symmetric under the exchange of z and z'.

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B. Jack polynomials

In two variables, Jack polynomials may be expressed in terms of Gegenbauer polynomials as,

$$P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi') = \frac{\lambda_-!}{(2\,\varepsilon)_{\lambda_-}}\,(\chi\chi')^{\frac{1}{2}(\lambda_1+\lambda_2)}C^{\,\varepsilon}_{\lambda_-}(s)\,, \quad \lambda_- = \lambda_1-\lambda_2\,, \quad s=\frac{\chi+\chi'}{2\,(\chi\chi')^{\frac{1}{2}}}\,, \tag{B.1}$$

where $\lambda_- = 0,1,2,\ldots$, and the normalisation here is such that $P^{(\varepsilon)}_{\lambda_1\lambda_2}(1,1)=1$.

We now summarize a few results regarding Jack polynomials which are used in the main text. The first few cases, for integer ε, are given by,

$$\begin{aligned}
P^{(0)}_{\lambda_1\lambda_2}(\chi,\chi') &= \frac{1}{2}\big(\chi^{\lambda_1}\chi'^{\lambda_2}+\chi^{\lambda_2}\chi'^{\lambda_1}\big)\,,\\
P^{(1)}_{\lambda_1\lambda_2}(\chi,\chi') &= \frac{1}{\lambda_-+1}\bigg(\frac{\chi^{\lambda_1+1}\chi'^{\lambda_2}-\chi^{\lambda_2}\chi'^{\lambda_1+1}}{\chi-\chi'}\bigg) = \frac{(\chi\chi')^{\lambda_2}}{\lambda_-+1}\bigg(\frac{\chi^{\lambda_-+1}-\chi'^{\lambda_-+1}}{\chi-\chi'}\bigg)\,,\\
P^{(2)}_{\lambda_1\lambda_2}(\chi,\chi') &= \frac{6}{\lambda_-+2}\bigg(\frac{\chi^{\lambda_1+3}\chi'^{\lambda_2}-\chi^{\lambda_2}\chi'^{\lambda_1+3}}{(\lambda_-+3)(\chi-\chi')^3}-\frac{\chi^{\lambda_1+2}\chi'^{\lambda_2+1}-\chi^{\lambda_2+1}\chi'^{\lambda_1+2}}{(\lambda_-+1)(\chi-\chi')^3}\bigg)\,,
\end{aligned} \tag{B.2}$$

and others may be obtained through use of the recurrence relation,

$$(\chi-\chi')^2P^{(\varepsilon+1)}_{\lambda_1\lambda_2}(\chi,\chi') = \frac{2(1+2\,\varepsilon)}{\lambda_-+1+\varepsilon}\Big(P^{(\varepsilon)}_{\lambda_1+2\,\lambda_2}(\chi,\chi')-P^{(\varepsilon)}_{\lambda_1+1\,\lambda_2+1}(\chi,\chi')\Big)\,, \tag{B.3}$$

following from standard identities for $C^{\varepsilon}_n(s)$.

We also have that (at least for $\varepsilon = 0,\frac{1}{2},1,\frac{3}{2},\ldots$)

$$C^{\varepsilon}_n(s) = (-1)^{2\varepsilon+1}C^{\varepsilon}_{-n-2\varepsilon}(s)\,, \tag{B.4}$$

which enables extension of Jack polynomials to some negative λ_-,

$$P^{(\varepsilon)}_{\lambda_1\,\lambda_2}(\chi,\chi') = P^{(\varepsilon)}_{\lambda_2-\varepsilon\,\lambda_1+\varepsilon}(\chi,\chi')\,. \tag{B.5}$$

Among useful identities are

$$(\chi\chi')^fP^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi') = P^{(\varepsilon)}_{\lambda_1+f\,\lambda_2+f}(\chi,\chi')\,, \tag{B.6}$$

and

$$(\chi+\chi')P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi') = \frac{\lambda_-+2\,\varepsilon}{\lambda_-+\varepsilon}\,P^{(\varepsilon)}_{\lambda_1+1\,\lambda_2}(\chi,\chi')+\frac{\lambda_-}{\lambda_-+\varepsilon}\,P^{(\varepsilon)}_{\lambda_1\,\lambda_2+1}(\chi,\chi')\,. \tag{B.7}$$

Defining,

$$D^{\pm}_n = \chi^n\,\frac{\partial}{\partial\chi}\pm\chi'^n\,\frac{\partial}{\partial\chi'}\,, \tag{B.8}$$

we have various derivative relations (which are used extensively in the main text)

$$D^{+}_0P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi') = \frac{(\lambda_1+\varepsilon)\lambda_-}{\lambda_-+\varepsilon}\,P^{(\varepsilon)}_{\lambda_1-1\,\lambda_2}(\chi,\chi')+\frac{\lambda_2(\lambda_-+2\,\varepsilon)}{\lambda_-+\varepsilon}\,P^{(\varepsilon)}_{\lambda_1\,\lambda_2-1}(\chi,\chi')\,,$$

$$
\begin{aligned}
D_2^+ P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi') &= \frac{\lambda_1(\lambda_- + 2\,\varepsilon)}{\lambda_- + \varepsilon}\, P^{(\varepsilon)}_{\lambda_1+1\,\lambda_2}(\chi,\chi') + \frac{(\lambda_2-\varepsilon)\lambda_-}{\lambda_- + \varepsilon}\, P^{(\varepsilon)}_{\lambda_1\,\lambda_2+1}(\chi,\chi')\,,\\
D_3^+ P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi') &= \frac{\lambda_1(\lambda_- + 2\,\varepsilon+1)(\lambda_- + 2\,\varepsilon)}{(\lambda_- + \varepsilon+1)(\lambda_- + \varepsilon)}\, P^{(\varepsilon)}_{\lambda_1+2\,\lambda_2}(\chi,\chi') -\\
&\quad -\varepsilon\,\frac{\lambda_-(\lambda_- + 2\,\varepsilon) + (\lambda_1+\lambda_2)(\varepsilon-1)}{(\lambda_- + \varepsilon-1)(\lambda_- + \varepsilon+1)}\, P^{(\varepsilon)}_{\lambda_1+1\,\lambda_2+1}(\chi,\chi') +\\
&\quad + \frac{(\lambda_2-\varepsilon)(\lambda_- -1)\lambda_-}{(\lambda_- + \varepsilon-1)(\lambda_- + \varepsilon)}\, P^{(\varepsilon)}_{\lambda_1\,\lambda_2+2}(\chi,\chi')\,, \qquad \text{(B.9)}
\end{aligned}
$$

with other such being obtained recursively.[33] Also, $P^{(\varepsilon)}_{\lambda_1\,\lambda_2}(\chi,\chi')$ is an eigenfunction of the Euler operator, D_1^+, with

$$
D_1^+\, P^{(\varepsilon)}_{\lambda_1\,\lambda_2}(\chi,\chi') = (\lambda_1+\lambda_2)\, P^{(\varepsilon)}_{\lambda_1\,\lambda_2}(\chi,\chi')\,, \qquad \text{(B.12)}
$$

and the following second order symmetric operator,

$$
D_\varepsilon^J = \chi^2 \frac{\partial^2}{\partial\chi^2} + \chi'^2 \frac{\partial^2}{\partial\chi'^2} + 2\,\varepsilon\, \frac{1}{\chi-\chi'} D_2^-\,, \qquad \text{(B.13)}
$$

with

$$
D_\varepsilon^J P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi') = \big(\lambda_1(\lambda_1 - 1 + 2\,\varepsilon) + \lambda_2(\lambda_2-1)\big) P^{(\varepsilon)}_{\lambda_1\lambda_2}(\chi,\chi')\,, \quad \lambda_1 \geq \lambda_2\,. \qquad \text{(B.14)}
$$

We may rewrite the operator in (5.4) as

$$
D_\varepsilon = \frac{\partial^2}{\partial\chi\partial\chi'} - \varepsilon \frac{1}{\chi-\chi'}\, D_0^- = \frac{1}{2\,\chi\,\chi'} \big(- D_\varepsilon^J + D_1^{+2} + (2\,\varepsilon+1)\, D_1^+ \big)\,, \qquad \text{(B.15)}
$$

so that

$$
D_\varepsilon\, P^{(\varepsilon)}_{\lambda_1\,\lambda_2}(\chi,\chi') = \lambda_2(\lambda_1+\varepsilon)\, P^{(\varepsilon)}_{\lambda_1-1\,\lambda_2-1}(\chi,\chi')\,, \qquad \text{(B.16)}
$$

which enables one to show (6.27). We also have the special identities, following from (B.3),

$$
\frac{1}{\chi-\chi'} D_0^-\, P^{(\varepsilon)}_{\lambda\,0}(\chi,\chi') = \frac{\lambda\,(\lambda-1)}{2\,(2\,\varepsilon+1)}\, P^{(\varepsilon+1)}_{\lambda-2\,0}(\chi,\chi')\,, \qquad \text{(B.17)}
$$

thus we have quite simply that,

$$
D_\varepsilon\, P^{(\varepsilon')}_{\lambda\,0}(\chi,\chi') = \frac{\lambda\,(\lambda-1)\,(\varepsilon'-\varepsilon)}{2\,(2\,\varepsilon'+1)}\, P^{(\varepsilon'+1)}_{\lambda-2\,0}(\chi,\chi') \qquad \text{(B.18)}
$$

which is used to prove (5.20).

[33] We may obtain other such derivative relations recursively using the trivial identity

$$
D_n^+ = (\chi+\chi')D_{n-1}^+ - \chi\chi'\, D_{n-2}^+\,, \qquad \text{(B.10)}
$$

along with (B.6) and (B.7). We may similarly find the action of,

$$
D_n^- = \frac{1}{\chi-\chi'}\,(2\,D_{n+1}^+ - (\chi+\chi')D_n^+)\,. \qquad \text{(B.11)}
$$

C. The $k=3$ case for $\mathrm{SO}(n)$ R symmetry

We here attempt to solve the Ward identities (4.31) for the $k=3$ case, showing agreement with the general k solutions. The Ward identities involve eight independent equations which we may conveniently take to be,

$$\sum_{i=0}^{3} \frac{1}{\chi^i} u^{i\varepsilon}\, \partial_\chi\, a_{i0} = 0\,, \quad \sum_{\substack{i,j\geq 0\\ i+j=3}} (1-\chi)^j\, v^{-j\varepsilon}\, \partial_\chi\, a_{ij} = 0\,, \quad \sum_{i=0}^{3} \Big(\frac{\chi-1}{\chi}\Big)^i \Big(\frac{u}{v}\Big)^{i\varepsilon} \partial_\chi\, a_{0i} = 0\,, \tag{C.1}$$

along with three other such with $\chi \leftrightarrow \chi'$ and,

$$\begin{aligned}\partial_\chi a_{11} + 2\frac{\chi^2}{(\chi-1)}\Big(\frac{v}{u^2}\Big)^\varepsilon \partial_\chi a_{00} + \frac{\chi}{\chi-1}\Big(\frac{v}{u}\Big)^\varepsilon \partial_\chi a_{10} + \chi\frac{1}{u^\varepsilon}\partial_\chi a_{01} + \\ + \frac{\chi-1}{\chi}\Big(\frac{u}{v}\Big)^\varepsilon \partial_\chi a_{12} - \frac{(\chi-1)^2}{\chi}\Big(\frac{u}{v^2}\Big)^\varepsilon \partial_\chi a_{03} = 0\,,\end{aligned} \tag{C.2}$$

involving a_{11}, along with another such with $\chi \leftrightarrow \chi'$. It is worth mentioning that any equation involving a_{11} necessarily involves at least six other of the a_{ij}.

It proves convenient to define

$$\begin{aligned} a_{03} &= v^{2\varepsilon}\Delta_\varepsilon\, uc\,,\\ a_{12} &= v^{2\varepsilon}\Delta_\varepsilon\, ub + v^\varepsilon\Delta_\varepsilon(\chi+\chi'-2)uc\,,\\ a_{02} &= v^{2\varepsilon}\Delta_\varepsilon\, ua + (uv)^\varepsilon\Delta_\varepsilon(\chi+\chi'-2u)c\,, \end{aligned} \tag{C.3}$$

where a, b, c are three arbitrary symmetric functions of χ, χ'. With these definitions we may use the operator identities (5.28) and (6.42) to find directly for the third equation in (C.1) that,

$$\partial_\chi\big(a_{00} - u^{2\varepsilon}\Delta_\varepsilon v\, a\big) + \frac{\chi-1}{\chi}\Big(\frac{u}{v}\Big)^\varepsilon \partial_\chi\big(a_{01} - (uv)^\varepsilon\Delta_\varepsilon(\chi+\chi'-2u)a - u^{2\varepsilon}\Delta_\varepsilon vc\big) = 0\,, \tag{C.4}$$

so that we may use the $k=1$ analysis to show that

$$\begin{aligned} a_{00} &= u^{2\varepsilon}\Delta_\varepsilon va + u^\varepsilon \sum_\lambda(-b_{1\,\lambda} + b_{1\,\lambda+1})\tilde P^{(\varepsilon)}_\lambda\,,\\ a_{01} &= (uv)^\varepsilon\Delta_\varepsilon(\chi+\chi'-2u)a + u^{2\varepsilon}\Delta_\varepsilon v\, c + v^\varepsilon\sum_\lambda b_{1\,\lambda}\,\tilde P^{(\varepsilon)}_\lambda\,, \end{aligned} \tag{C.5}$$

for arbitrary $b_{1\,\lambda}$. Similarly, we may find for the second equation in (C.1) that,

$$\partial_\chi\big(a_{30} - \Delta_\varepsilon\, uvb\big) + \frac{1}{1-\chi}v^\varepsilon\partial_\chi\big(a_{21} - v^\varepsilon\Delta_\varepsilon(\chi+\chi'-2)u\, b - \Delta_\varepsilon uv\, c\big) = 0\,, \tag{C.6}$$

so that

$$a_{21} = v^\varepsilon\Delta_\varepsilon(\chi+\chi'-2)u\, b + \Delta_\varepsilon uv\, c + v^\varepsilon\sum_\lambda b_{3\,\lambda}\,\tilde P^{(\varepsilon)}_\lambda\,,$$

$$a_{30} = \Delta_\varepsilon\, uvb + \sum_\lambda \big(b_{3\,\lambda-1} - b_{3\,\lambda}\big)\tilde{P}^{(\varepsilon)}_\lambda\,, \tag{C.7}$$

for arbitrary $b_{3\,\lambda}$. Using (5.28) and (6.42) and also the identity

$$\partial_\chi u^\varepsilon\, \tilde{P}^{(\varepsilon)}_\lambda(\chi,\chi') = \frac{1}{\chi} u^\varepsilon \partial_\chi\, \tilde{P}^{(\varepsilon)}_{\lambda+1}(\chi,\chi')\,, \tag{C.8}$$

with (C.7) and (C.5), we may easily deal with the first equation in (C.1) to find that

$$\begin{aligned}\partial_\chi\bigg(a_{10} + u^\varepsilon\Delta_\varepsilon(\chi+\chi')v\, a - u^{2\varepsilon}\Delta_\varepsilon v\, b - \sum_\lambda (b_{1\,\lambda-1} - b_{1\,\lambda})\tilde{P}^{(\varepsilon)}_\lambda\bigg) + \\ +\frac{1}{\chi}u^\varepsilon\partial_\chi\bigg(a_{20} - \Delta_\varepsilon uv\, a + u^\varepsilon\Delta_\varepsilon(\chi+\chi')v\, b + u^\varepsilon\sum_\lambda (b_{3\,\lambda} - b_{3\,\lambda+1})\tilde{P}^{(\varepsilon)}_\lambda\bigg) = 0\,. \end{aligned}\tag{C.9}$$

This may be solved similarly to the $k=1$ case and we find that

$$\begin{aligned} a_{10} &= -u^\varepsilon\Delta_\varepsilon(\chi+\chi')v\, a + u^{2\varepsilon}\Delta_\varepsilon v\, b + \sum_\lambda \big(b_{1\,\lambda-1} - b_{1\,\lambda} + u^\varepsilon a_\lambda\big)\tilde{P}^{(\varepsilon)}_\lambda\,, \\ a_{20} &= \Delta_\varepsilon uv\, a - u^\varepsilon\Delta_\varepsilon(\chi+\chi')v\, b + \sum_\lambda \big(u^\varepsilon(-b_{3\,\lambda} + b_{3\,\lambda+1}) - a_{\lambda-1}\big)\tilde{P}^{(\varepsilon)}_\lambda\,, \end{aligned}\tag{C.10}$$

with arbitrary a_λ.

More difficult to deal with is (C.2). We first deal with each of the two variable contributions in turn. Using (5.28) and (6.42) we may find that

$$\begin{aligned} 2\frac{\chi^2}{\chi-1}\Big(\frac{v}{u^2}\Big)^\varepsilon\partial_\chi u^{2\varepsilon}\Delta_\varepsilon v\, a - \frac{\chi}{\chi-1}\Big(\frac{v}{u}\Big)^\varepsilon\partial_\chi\, u^\varepsilon\Delta_\varepsilon(\chi+\chi')v\, a + \\ +\chi\frac{1}{u^\varepsilon}\partial_\chi(uv)^\varepsilon\Delta_\varepsilon(\chi+\chi'-2u)a = -\partial_x v^\varepsilon\Delta_\varepsilon(\chi+\chi'-2)u\, a\,, \end{aligned}\tag{C.11}$$

for the a contributions coming from a_{00}, a_{10}, a_{01}. Similarly we may find that

$$\frac{\chi}{\chi-1}\Big(\frac{v}{u}\Big)^\varepsilon\partial_\chi u^{2\varepsilon}\Delta_\varepsilon v\, b + \frac{\chi-1}{\chi}\Big(\frac{u}{v}\Big)^\varepsilon\partial_\chi v^{2\varepsilon}\Delta_\varepsilon u\, b = -\partial_\chi(uv)^\varepsilon\Delta_\varepsilon\,(\chi+\chi'-2u)b\,, \tag{C.12}$$

for the b contributions coming from a_{10}, a_{12}. Similarly we may also find that

$$\begin{aligned} \chi\frac{1}{u^\varepsilon}\partial_\chi u^{2\varepsilon}\Delta_\varepsilon v\, c + \frac{\chi-1}{\chi}\Big(\frac{u}{v}\Big)^\varepsilon\partial_\chi v^\varepsilon\Delta_\varepsilon(\chi+\chi'-2)u\, c - \\ -\frac{(\chi-1)^2}{\chi}\Big(\frac{u}{v^2}\Big)^\varepsilon\partial_\chi v^{2\varepsilon}\Delta_\varepsilon u\, c = \partial_\chi u^\varepsilon\Delta_\varepsilon(\chi+\chi')v\, c\,, \end{aligned}\tag{C.13}$$

for the c contributions from a_{01}, a_{12}, a_{03}. The only other contributions to (C.2) come from $b_{1\,\lambda}$ and a_λ, however, due to

$$\begin{aligned} 2\frac{\chi^2}{(\chi-1)}\Big(\frac{v}{u^2}\Big)^\varepsilon\partial_\chi\, u^\varepsilon\big(\tilde{P}^{(\varepsilon)}_{\lambda-1}(\chi,\chi') - \tilde{P}^{(\varepsilon)}_\lambda(\chi,\chi')\big) + \\ +\frac{\chi}{\chi-1}\Big(\frac{v}{u}\Big)^\varepsilon\partial_\chi\big(\tilde{P}^{(\varepsilon)}_{\lambda+1}(\chi,\chi') - \tilde{P}^{(\varepsilon)}_\lambda(\chi,\chi')\big) = -\chi\frac{1}{u^\varepsilon}\,\partial_\chi\, v^\varepsilon\tilde{P}^{(\varepsilon)}_\lambda(\chi,\chi')\,, \end{aligned}\tag{C.14}$$

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those from $b_{1\,\lambda}$ vanish. We thus find from (C.2) that

$$\partial_\chi \sum_\lambda a_\lambda \, \tilde{P}^{(\varepsilon)}_\lambda + \frac{\chi-1}{\chi}\Big(\frac{u}{v}\Big)^\varepsilon \partial_\chi \big(a_{11} - v^\varepsilon \, \Delta_\varepsilon \, (\chi+\chi'-2)\, u\, a - \\ -(u\,v)^\varepsilon \, \Delta_\varepsilon \, (x+\chi'-2u)\, b + u^\varepsilon \, \Delta_\varepsilon \, v\, (\chi+\chi')c\big) \;=0\,, \tag{C.15}$$

so that the solution is given by

$$a_{11} \;= v^\varepsilon \, \Delta_\varepsilon \, (\chi+\chi'-2)\, u\, a + (u\,v)^\varepsilon \, \Delta_\varepsilon \, (\chi+\chi'-2u)\, b - \\ -u^\varepsilon \, \Delta_\varepsilon \, v\, (\chi+\chi')c + v^\varepsilon \sum_\lambda b_{2\,\lambda} \, \tilde{P}^{(\varepsilon)}_\lambda \,, \tag{C.16}$$

where

$$a_\lambda = -b_{2\,\lambda} + b_{2\,\lambda+1}\,. \tag{C.17}$$

With the solutions (C.7), (C.5), (C.10) and (C.16) we may then find that

$$G^{(\varepsilon)}_3(\chi,\chi';\alpha,\alpha') \;= u^{2\varepsilon}\, \Delta_\varepsilon \, (\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)a + \\ +u^{3\varepsilon}\, \alpha\alpha' \, \Delta_\varepsilon \, (\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)b + \\ +\Big(\frac{u^3}{v}\Big)^\varepsilon \, (\alpha-1)(\alpha'-1)\, \Delta_\varepsilon \, (\chi\alpha-1)(\chi'\alpha-1)(\chi\alpha'-1)(\chi'\alpha'-1)c + \\ +\sum_{n=1}^{3} u^{n\varepsilon}(\alpha\alpha')^{n-1}\, \mathcal{H}^{(\varepsilon)}_n(\chi,\chi';\alpha,\alpha')\,, \tag{C.18}$$

which agrees with the general k analysis. Given the comments on the $k=2$ case, for $\varepsilon \neq 1$ we may absorb the $\mathcal{H}^{(\varepsilon)}_n$, $n=2,3$ contributions to (C.18).

D. Recurrence relations for conformal partial waves

To perform the conformal partial wave expansion of (7.10) requires other results than those mentioned in [22] essentially because, whereas recurrence relations for $(v\pm1)\mathcal{G}^{(\ell)}_\Delta/u$ were found there (which are of direct relevance to four dimensions, see [11]), here matters are very much more complicated by the presence of the operator Δ_ε. Here we need results of the form e.g. $u^{2\varepsilon}\Delta_\varepsilon(v\pm1)ua$, for a given by (7.11).

In order to simplify the potentially very long winded recurrence relations for conformal partial waves, we define

$$F_{r,s}(\chi,\chi') = D(\Delta,\ell,r,s)\mathcal{G}^{(\ell+r-s)}_{\Delta+r+s}(\chi,\chi')\,, \tag{D.1}$$

where r,s are integers, and $D(\ell,\Delta,r,s)$ is a constant so that

$$D(\Delta,\ell,r,s) = 2^{r-s}\; B^{\mathrm{sgn}\,r}_{\frac12(\Delta+\ell),r}\; B^{\mathrm{sgn}\,s}_{\frac12(\Delta-\ell)-\varepsilon,s} \times \\ \times A_{\ell+1,-\frac12(\mathrm{sgn}(r-s)-1)(r-s)}\; A_{2-\Delta,-\frac12(\mathrm{sgn}(r+s)+1)(r+s)}\,, \tag{D.2}$$

where $D(\Delta,\ell,0,0)\equiv 1$ and

$$A_{\lambda,t} = \frac{(\lambda+\varepsilon)_t(\lambda+\varepsilon-1)_t}{(\lambda)_t(\lambda+2\varepsilon-1)_t}\,,\quad B^+_{\lambda,t} = 16^{-t}\frac{(\lambda)_t(\lambda+\varepsilon-1)_t}{(\lambda-\frac12)_t(\lambda+\frac12)_t}\,,\quad B^-_{\lambda,t} = \frac{(\lambda)_t}{(\lambda+1-\varepsilon)_t}\,. \tag{D.3}$$

We have trivially that, recalling that $\alpha_\pm = \frac{1}{2}(\Delta \pm \ell) - 2\varepsilon$,

$$u^{2\varepsilon}\Delta_\varepsilon \sum_{m,n\geq 0} r_{mn}(\Delta,\ell)(\mathcal{E}^{(\varepsilon)}_{\alpha_+ + m\,\alpha_- + n})^{-1}\; P^{(\varepsilon)}_{\alpha_+ + m\,\alpha_- + n}(\chi,\chi') = F_{0,0}\,. \tag{D.4}$$

We may also find that,

$$\begin{aligned} u^{2\varepsilon}\Delta_\varepsilon \sum_{m,n\geq 0} r_{mn}(\Delta,\ell)(\mathcal{E}^{(\varepsilon)}_{\alpha_+ + m\,\alpha_- + n})^{-1}\,\frac{(v-1)}{u}\, P^{(\varepsilon)}_{\alpha_+ + m\,\alpha_- + n}(\chi,\chi') = \\ = F_{-1,0} + F_{0,-1} + F_{0,1} + F_{1,0}\,, \end{aligned} \tag{D.5}$$

which is useful for expanding $\Delta_\varepsilon u\,(v-1)a$, from which (7.23) follows. We may also determine that,

$$\begin{aligned} &\frac{1}{2}u^{2\varepsilon}\Delta_\varepsilon \sum_{m,n\geq 0} r_{mn}(\Delta,\ell)(\mathcal{E}^{(\varepsilon)}_{\alpha_+ + m\,\alpha_- + n})^{-1}\,\frac{(v+1)}{u}\, P^{(\varepsilon)}_{\alpha_+ + m\,\alpha_- + n}(\chi,\chi') = \\ &\quad = F_{-1,-1} + F_{-1,1} + F_{1,-1} + F_{1,1} + \\ &\qquad + \frac{1}{4}\Big(1 - \frac{1}{2}\varepsilon(\varepsilon-1)\big(\mathcal{A}_{\ell+1} + \mathcal{A}_{2-\Delta} - (2\varepsilon-1)(2\varepsilon-3)\mathcal{A}_{\ell+1}\mathcal{A}_{2-\Delta}\big)\Big)F_{0,0}\,, \end{aligned} \tag{D.6}$$

where

$$\mathcal{A}_\lambda = \frac{1}{(\lambda+\varepsilon)(\lambda+\varepsilon-2)}\,, \tag{D.7}$$

which is useful for expanding $\Delta_\varepsilon u(v+1)a$. Useful for expanding $\Delta_\varepsilon(v-1)^2 a$ is the following, namely,

$$\begin{aligned} &u^{2\varepsilon}\Delta_\varepsilon \sum_{m,n\geq 0} r_{mn}(\Delta,\ell)(\mathcal{E}^{(\varepsilon)}_{\alpha_+ + m\,\alpha_- + n})^{-1}\,\frac{(v-1)^2}{u^2}\, P^{(\varepsilon)}_{\alpha_+ + m\,\alpha_- + n}(\chi,\chi') = \\ &\quad = F_{-2,0} + F_{0,-2} + F_{0,2} + F_{2,0} + \\ &\qquad + 2(1 - \varepsilon(\varepsilon-1)\mathcal{A}_{\ell+1})\big(F_{-1,-1} + F_{1,1}\big) + 2(1 - \varepsilon(\varepsilon-1)\mathcal{A}_{2-\Delta})\big(F_{-1,1} + F_{1,-1}\big) + \\ &\qquad + C_{\Delta,\ell}F_{0,0}\,, \end{aligned} \tag{D.8}$$

where

$$\begin{aligned} 4C_{\Delta,\ell} &= 1 - \frac{1}{2}(2\varepsilon-1)(2\varepsilon-3)\big(\mathcal{B}_{\ell+\varepsilon+1-\Delta} + \mathcal{B}_{\Delta+\ell-\varepsilon-1}\big) + \\ &\quad + \varepsilon(\varepsilon-1)\big(\varepsilon(\varepsilon-1)\mathcal{A}_{\ell+1}\,\mathcal{A}_{2-\Delta} - \mathcal{A}_{\ell+1} - \mathcal{A}_{2-\Delta}\big) \times \\ &\quad \times \bigg(\frac{1}{2} - (2\varepsilon-1)(2\varepsilon-3)\Big(\frac{1}{4}\mathcal{B}_{\ell+\varepsilon+1-\Delta} + \frac{1}{4}\mathcal{B}_{\Delta+\ell-\varepsilon-1} - 3\mathcal{B}_{\ell+\varepsilon+1-\Delta}\mathcal{B}_{\Delta+\ell-\varepsilon-1}\Big)\bigg)\,, \end{aligned} \tag{D.9}$$

where

$$\mathcal{B}_\lambda = \frac{1}{(\lambda+\varepsilon+2)(\lambda+\varepsilon-2)}\,. \tag{D.10}$$

Useful for expanding $\Delta_\varepsilon(v-1)(v+1)a$ is the following,

$$\frac{1}{2}u^{2\varepsilon}\Delta_\varepsilon \sum_{m,n\geq 0} r_{mn}(\Delta,\ell)(\mathcal{E}^{(\varepsilon)}_{\alpha_+ + m\,\alpha_- + n})^{-1}\,\frac{v^2-1}{u^2}\, P^{(\varepsilon)}_{\alpha_+ + m\,\alpha_- + n}(\chi,\chi')$$

$$
\begin{aligned}
&= F_{-2,-1} + F_{-1,-2} + F_{-2,1} + F_{1,-2} + F_{-1,2} + F_{2,-1} + F_{1,2} + F_{2,1} + \\
&\quad + a_{\Delta,\ell} F_{0,-1} + b_{\Delta,\ell} F_{-1,0} + c_{\Delta,\ell} F_{1,0} + d_{\Delta,\ell} F_{0,1} \,,
\end{aligned} \tag{D.11}
$$

where

$$
b_{\Delta,\ell} = a_{\Delta,-\ell-2\varepsilon} \,, \qquad c_{\Delta,\ell} = a_{2\varepsilon+2-\Delta,\ell} \,, \qquad d_{\Delta,\ell} = a_{2\varepsilon+2-\Delta,-\ell-2\varepsilon} \,, \tag{D.12}
$$

with

$$
\begin{aligned}
4a_{\Delta,\ell} &= 3 - \frac{3}{2}\varepsilon(\varepsilon-1)(\mathcal{C}_{\ell+1} + \mathcal{C}_{2-\Delta}) - (2\varepsilon-1)(2\varepsilon-3)\mathcal{B}_{\Delta+\ell-\varepsilon-1} + \\
&\quad + \varepsilon(\varepsilon-1)(2\varepsilon-1)(2\varepsilon-3) \times \\
&\quad \times \big(\mathcal{C}_{\ell+1}\mathcal{C}_{2-\Delta} + \frac{1}{2}\mathcal{B}_{\Delta+\ell-\varepsilon-1}(\mathcal{C}_{\ell+1} + \mathcal{C}_{2-\Delta} - 10\mathcal{C}_{\ell+1}\mathcal{C}_{2-\Delta})\big) \,,
\end{aligned} \tag{D.13}
$$

where

$$
\mathcal{C}_\lambda = \frac{1}{(\lambda+\varepsilon+1)(\lambda+\varepsilon-2)} \,. \tag{D.14}
$$

Useful for expanding $\Delta_\varepsilon (v+1)^2 a$ is the following

$$
\begin{aligned}
&\frac{1}{4}u^{2\varepsilon}\Delta_\varepsilon \sum_{m,n\geq 0} r_{mn}(\Delta,\ell)(\mathcal{E}^{(\varepsilon)}_{\alpha_+ + m\,\alpha_- + n})^{-1}\,\frac{(v+1)^2}{u^2}\,P^{(\varepsilon)}_{\alpha_+ + m\,\alpha_- + n}(\chi,\chi') = \\
&\qquad = F_{-2,-2} + F_{-2,2} + F_{2,-2} + F_{2,2} + \\
&\qquad\quad + \frac{1}{8}(1 - (2\varepsilon-1)(2\varepsilon-3)\mathcal{B}_{\ell+\varepsilon+1-\Delta})(F_{-2,0} + F_{2,0}) + \\
&\qquad\quad + \frac{1}{8}(1 - (2\varepsilon-1)(2\varepsilon-3)\mathcal{B}_{\Delta+\ell-\varepsilon-1})(F_{0,-2} + F_{0,2}) + \\
&\qquad\quad + e_{\Delta,\ell}F_{-1,-1} + f_{\Delta,\ell}F_{-1,1} + g_{\Delta,\ell}F_{1,-1} + h_{\Delta,\ell}F_{1,1} + D_{\Delta,\ell}F_{0,0} \,,
\end{aligned} \tag{D.15}
$$

where

$$
f_{\Delta,\ell} = e_{-\ell-3,1-\Delta} \,, \qquad g_{\Delta,\ell} = e_{-\ell-1,1-\Delta} \,, \qquad h_{\Delta,\ell} = e_{2+\Delta,\ell} \,, \tag{D.16}
$$

with

$$
2\,e_{\Delta,\ell} = 1 - \frac{1}{2}\varepsilon(\varepsilon-1)\big(\mathcal{A}_{\ell+1} + \mathcal{B}_{2-\Delta} - (2\varepsilon-1)(2\varepsilon-3)\mathcal{A}_{\ell+1}\mathcal{B}_{2-\Delta}\big) \,. \tag{D.17}
$$

The important point here is that from these results we may show compatibility of (7.10) with (7.14) due to the correspondence

$$
F_{r,s} \leftrightarrow (\Delta + r + s)_{\ell+r-s} \,, \tag{D.18}
$$

between the conformal partial wave and an operator in the OPE.

E. Representation theory

Following an approach like that for construction of superconformal multiplets in [37] based on using the Racah-Speiser algorithm (described in [38]), we show how we may verify that the conformal partial wave expansion results for long multiplets are consistent with representation theory. Generically we have that the states appearing have $\mathrm{SO}(d-1,1) \otimes \mathrm{SO}(n)$ eigenvalues given by unconstrained tensor products of the form

$$\prod_{I,A} (Q_{IA})^{n_{IA}} |\mathrm{h};\mathrm{k}\rangle^{\mathrm{hw}}\,, \quad n_{IA} \in \{0,1\}\,, \tag{E.1}$$

where $|\mathrm{h};\mathrm{k}\rangle^{\mathrm{hw}}$ is a highest weight state, annihilated by all generators of the appropriate superconformal algebra, apart from the dilatation operator, for which it has eigenvalue $i\Delta$, the momentum operators, the super-charges Q_{IA}, where I, A are, respectively, $\mathrm{SO}(d-1,1)$, $\mathrm{SO}(n)$ indices, the lowering operators in the Chevalley-Serre basis of the algebra for $\mathrm{SO}(d-1,1) \otimes \mathrm{SO}(n)$ and the generators of the Cartan sub-algebra for which it has respective eigenvalues h, k.

The $\mathrm{SO}(d-1,1) \otimes \mathrm{SO}(n)$ Cartan sub-algebra eigenvalues of Q_{IA} are given by those in the weight system for some fundamental representation of the group, which we denote generically by $[\mathrm{i}_I;\mathrm{j}_A]$. Thus, the states in (E.1) have h', k' eigenvalues and conformal dimensions Δ' given by,

$$[\mathrm{h}';\mathrm{k}'] = \sum_{I,A} n_{IA}[\mathrm{i}_I;\mathrm{j}_A] + [\mathrm{h};\mathrm{k}]\,, \qquad \Delta' = \Delta + \frac{1}{2}\sum_{I,A} n_{IA}\,. \tag{E.2}$$

Specific eigenvalues h, k requires use of the Racah-Speiser algorithm when we may have cancellations among representations for fixed Δ'.

For example, for $d = 3$ and $n = 8$ we have that Q_{IA} belongs to the $[\frac{1}{2}]$ representation of $\mathrm{SO}(2,1)$ and the $[1,0,0,0]$ representation of $\mathrm{SO}(8)$, and thus that

$$\mathrm{i}_I \in \left\{ \left[\frac{1}{2}\right], \left[-\frac{1}{2}\right] \right\}, \tag{E.3}$$

and

$$\begin{aligned} \mathrm{j}_A \in \{&[1,0,0,0],\ [-1,1,0,0],\ [0,-1,1,1],\ [0,0,-1,1],\\ &[0,0,1,-1],\ [0,1,-1,-1],\ [1,-1,0,0],\ [-1,0,0,0]\}\,, \end{aligned} \tag{E.4}$$

whereby we may easily work out all possibilities for h', k', Δ' in (E.2). For application to the Racah-Speiser algorithm we may note the action of the generators of the appropriate Weyl groups on generic weights; $[h]^\pi = [-h-1]$ for $\mathrm{SO}(2,1)$ and

$$\begin{aligned} [k_1,k_2,k_3,k_4]^{\pi_1} &= [-k_1-2, k_1+k_2+1, k_3, k_4]\,,\\ [k_1,k_2,k_3,k_4]^{\pi_2} &= [k_1, k_2+k_3+1, -k_3-2, k_4]\,,\\ [k_1,k_2,k_3,k_4]^{\pi_3} &= [k_1, k_2+k_4+1, k_3, -k_4-2]\,,\\ [k_1,k_2,k_3,k_4]^{\pi_4} &= [k_1+k_2+1, -k_2-2, k_2+k_3+1, k_2+k_4+1] \end{aligned} \tag{E.5}$$

for $\mathrm{SO}(8)$. We may thus verify that (7.14) is compatible with states appearing in a long superconformal multiplet starting from a state with conformal dimension $\Delta' = \Delta - 4$, with spin $\mathrm{h} = [\ell]$ and being an $\mathrm{SO}(8)$ singlet, $\mathrm{k} = [0,0,0,0]$.

For $d = 6$ and $n = 5$ then

$$\mathrm{i}_I \in \{[1,0,0],\ [-1,1,0],\ [0,-1,1],\ [0,0,-1]\}\,, \tag{E.6}$$

and

$$\mathrm{j}_A \in \{[1,0],\ [-1,1],\ [1,-1],\ [-1,0]\}\,. \tag{E.7}$$

The action of the generators of the appropriate Weyl groups is given by

$$\begin{aligned}
[h_1,h_2,h_3]^{\pi_1} &= [-h_1-2, h_1+h_2+1, h_3]\,,\\
[h_1,h_2,h_3]^{\pi_2} &= [h_1+h_2+1, -h_2-2, h_2+h_3+1]\,,\\
[h_1,h_2,h_3]^{\pi_3} &= [h_1, h_2+h_3+1, -h_3-2]\,,
\end{aligned} \tag{E.8}$$

for $\mathrm{SO}(5,1)$ and

$$[k_1,k_2]^{\rho_1} = [-k_1-2, k_1+k_2+1]\,, \qquad [k_1,k_2]^{\rho_2} = [k_1+2k_2+2, -k_2-2]\,, \tag{E.9}$$

for $\mathrm{SO}(5)$. Again we may verify that (7.14) is compatible with states appearing in a long superconformal multiplet starting from a state with conformal dimension $\Delta' = \Delta - 4$, with spin $\mathrm{h} = [0,\ell,0]$ and being an $\mathrm{SO}(5)$ singlet, $\mathrm{k} = [0,0]$.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200];
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[2] E. Witten, *Some comments on string dynamics*, contributed to *STRINGS 95: future perspectives in string theory*, Los Angeles, 13-18 Mar 1995, in *Los Angeles 1995: future perspectives in string theory*, pp. 501–523 [hep-th/9507121];
A. Strominger, *Open p-branes*, *Phys. Lett.* **B 383** (1996) 44 [hep-th/9512059];
N. Seiberg, *Notes on theories with 16 supercharges*, *Nucl. Phys.* **67** *(Proc. Suppl.)* (1998) 158 [hep-th/9705117].

[3] M. Cvetic and D. Youm, *Phys. Rev.* **D 53** (1996) 5400;
S. Ferrara and R. Kallosh, *Universality of supersymmetric attractors*, *Phys. Rev.* **D 54** (1996) 1525 [hep-th/9603090];
H. Lu, C.N. Pope and K.S. Stelle, *Multiplet structures of BPS solitons*, *Class. and Quant. Grav.* **15** (1998) 537 [hep-th/9708109];
V. Balasubramanian, F. Larsen and R.G. Leigh, *Branes at angles and black holes*, *Phys. Rev.* **D 57** (1998) 3509 [hep-th/9704143];
S. Ferrara and J.M. Maldacena, *Branes, central charges and u-duality invariant BPS conditions*, *Class. and Quant. Grav.* **15** (1998) 749 [hep-th/9706097];
S. Ferrara and M. Günaydin, *Orbits of exceptional groups, duality and BPS states in string theory*, *Int. J. Mod. Phys.* **A 13** (1998) 2075 [hep-th/9708025];
B. de Wit and H. Nicolai, $N = 8$ *supergravity with local* $\mathrm{SO}(8) \times \mathrm{SU}(8)$ *invariance*, *Phys. Lett.* **B 108** (1982) 285.

JHEP09(2004)056

[4] B.U. Eden, P.S. Howe, A. Pickering, E. Sokatchev and P.C. West, *Four-point functions in $N=2$ superconformal field theories*, *Nucl. Phys.* **B 581** (2000) 523 [hep-th/0001138].

[5] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained $N=2$ matter, Yang-Mills and supergravity theories in harmonic superspace*, *Class. and Quant. Grav.* **1** (1984) 469; *Unconstrained off-shell $N=3$ supersymmetric Yang-Mills theory*, *Class. and Quant. Grav.* **2** (1985) 155;
A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, *Harmonic superspace*, Cambridge University Press, Cambridge 2001.

[6] P.S. Howe and G.G. Hartwell, *A superspace survey*, *Class. and Quant. Grav.* **12** (1995) 1823; *(n,p,q) harmonic superspace*, *Int. J. Mod. Phys.* **A 10** (1995) 3901 [hep-th/9412147].

[7] B. Eden, A.C. Petkou, C. Schubert and E. Sokatchev, *Partial non-renormalisation of the stress-tensor four-point function in $N=4$ SYM and AdS/CFT*, *Nucl. Phys.* **B 607** (2001) 191 [hep-th/0009106].

[8] K.A. Intriligator, *Bonus symmetries of $N=4$ super–Yang-Mills correlation functions via AdS duality*, *Nucl. Phys.* **B 551** (1999) 575 [hep-th/9811047].

[9] G. Arutyunov, S. Frolov and A.C. Petkou, *Operator product expansion of the lowest weight CPOS in $N=4$ SYM_4 at strong coupling*, *Nucl. Phys.* **B 586** (2000) 547 [hep-th/0005182]; *Perturbative and instanton corrections to the ope of cpos in $N=4$ SYM_4*, *Nucl. Phys.* **B 602** (2001) 238 [hep-th/0010137].

[10] B. Eden, C. Schubert and E. Sokatchev, *Three-loop four-point correlator in $N=4$ SYM*, *Phys. Lett.* **B 482** (2000) 309 [hep-th/0003096]; *Four-point functions of chiral primary operators in $N=4$ SYM*, hep-th/0010005;
M. Bianchi, S. Kovacs, G. Rossi and Y.S. Stanev, *Anomalous dimensions in $N=4$ SYM theory at order g^4*, *Nucl. Phys.* **B 584** (2000) 216 [hep-th/0003203];
G. Arutyunov, B. Eden, A.C. Petkou and E. Sokatchev, *Exceptional non-renormalization properties and ope analysis of chiral four-point functions in $N=4$ SYM_4*, *Nucl. Phys.* **B 620** (2002) 380 [hep-th/0103230].

[11] F.A. Dolan and H. Osborn, *Superconformal symmetry, correlation functions and the operator product expansion*, *Nucl. Phys.* **B 629** (2002) 3 [hep-th/0112251].

[12] P.J. Heslop and P.S. Howe, *Four-point functions in $N=4$ SYM*, *J. High Energy Phys.* **01** (2003) 043 [hep-th/0211252].

[13] G. Arutyunov, F.A. Dolan, H. Osborn and E. Sokatchev, *Correlation functions and massive Kaluza-Klein modes in the AdS/CFT correspondence*, *Nucl. Phys.* **B 665** (2003) 273 [hep-th/0212116].

[14] G. Arutyunov, S. Penati, A. Santambrogio and E. Sokatchev, *Four-point correlators of BPS operators in $N=4$ SYM at order g^4*, *Nucl. Phys.* **B 670** (2003) 103 [hep-th/0305060].

[15] M. Nirschl and H. Osborn, *Superconformal Ward identities and their solution*, hep-th/0407060.

[16] G. Arutyunov and E. Sokatchev, *Implications of superconformal symmetry for interacting (2,0) tensor multiplets*, *Nucl. Phys.* **B 635** (2002) 3 [hep-th/0201145].

[17] P.J. Heslop, *Superconformal field theories in analytic superspace*, hep-th/0403144.

[18] A. Galperin, E. Ivanov and V. Ogievetsky, *Grassmann analiticity and extended supersymmetries*, *Sov. Phys. JETP Lett.* **33** (1981) 176.

JHEP09(2004)056

[19] P. Heslop and P.S. Howe, *On harmonic superspaces and superconformal fields in four dimensions*, *Class. and Quant. Grav.* **17** (2000) 3743 [hep-th/0005135].

[20] S. Ferrara and E. Sokatchev, *Superconformal interpretation of BPS states in AdS geometries*, *Int. J. Theor. Phys.* **40** (2001) 935 [hep-th/0005151].

[21] H. Jack, *A class of symmetric polynomials with a parameter*, *Proc. Roy. Soc. Edinburgh*, **69** (1970) 1.

[22] F.A. Dolan and H. Osborn, *Conformal partial waves and the operator product expansion*, *Nucl. Phys.* **B 678** (2004) 491 [hep-th/0309180].

[23] G. Mack and A. Salam, *Finite component field representations of the conformal group*, *Ann. Phys. (NY)* **53** (1969) 174.

[24] E.T. Whitteker and G.N. Watson, *A course of modern analysis*, Cambridge University Press, Cambridge 1927.

[25] I.T. Todorov and R.P. Zaikov, *Spectral representation of the covariant two-point function and infinite-component fields with arbitrary mass spectrum*, *J. Math. Phys.* **10** (1969) 2014.

[26] V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova and I.T. Todorov, *Harmonic analysis on the n-dimensional Lorentz group*, Lecture Notes in Physics **63**, Springer-Verlag 1977.

[27] F.A. Dolan and H. Osborn, *Implications of $N=1$ superconformal symmetry for chiral fields*, *Nucl. Phys.* **B 593** (2001) 599 [hep-th/0006098].

[28] N.I. Usyukina and A.I. Davydychev, *An approach to the evaluation of three and four point ladder diagrams*, *Phys. Lett.* **B 298** (1993) 363.

[29] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT_d/AdS_{d+1} correspondence*, *Nucl. Phys.* **B 546** (1999) 96 [hep-th/9804058];
S.-M. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three-point functions of chiral operators in $D=4$, $N=4$ SYM at large-N*, *Adv. Theor. Math. Phys.* **2** (1998) 697 [hep-th/9806074];
P.S. Howe, E. Sokatchev and P.C. West, *3-point functions in $N=4$ Yang-Mills*, *Phys. Lett.* **B 444** (1998) 341 [hep-th/9808162].

[30] P.S. Howe and P.C. West, *Non-perturbative Green's functions in theories with extended superconformal symmetry*, *Int. J. Mod. Phys.* **A 14** (1999) 2659 [hep-th/9509140];
P.S. Howe and P.C. West, *Is $N=4$ Yang-Mills theory soluble?*, hep-th/9611074.

[31] J.-H. Park, *$N=1$ superconformal symmetry in 4-dimensions*, *Int. J. Mod. Phys.* **A 13** (1998) 1743 [hep-th/9703191];
H. Osborn, *$N=1$ superconformal symmetry in four-dimensional quantum field theory*, *Ann. Phys. (NY)* **272** (1999) 243 [hep-th/9808041].

[32] J.-H. Park, *Superconformal symmetry in six-dimensions and its reduction to four*, *Nucl. Phys.* **B 539** (1999) 599 [hep-th/9807186]; *Superconformal symmetry and correlation functions*, *Nucl. Phys.* **B 559** (1999) 455 [hep-th/9903230]; *Superconformal symmetry in three-dimensions*, *J. Math. Phys.* **41** (2000) 7129 [hep-th/9910199].

[33] P.S. Howe, *On harmonic superspace*, hep-th/9812133;
Aspects of the $D=6$, (2,0) tensor multiplet, *Phys. Lett.* **B 503** (2001) 197 [hep-th/0008048].

[34] B. Eden, S. Ferrara and E. Sokatchev, *(2,0) superconformal OPEs in $D=6$, selection rules and non-renormalization theorems*, *J. High Energy Phys.* **11** (2001) 020 [hep-th/0107084].

[35] S. Ferrara and E. Sokatchev, *Universal properties of superconformal opes for 1/2 BPS operators in* $3 \leq d \leq 6$, *New J. Phys.* **4** (2002) 2 [hep-th/0110174].

[36] F.A. Dolan, *Aspects of superconformal quantum field theory*, Ph.D. Thesis 2003, unpublished.

[37] F.A. Dolan and H. Osborn, *On short and semi-short representations for four dimensional superconformal symmetry*, *Ann. Phys. (NY)* **307** (2003) 41 [hep-th/0209056].

[38] J. Fuchs and C. Schweigert, *Symmetries, Lie algebras and representations*, Cambridge University Press, Cambridge 1997.

[39] P.J. Heslop, *Aspects of superconformal field theories in six dimensions*, *J. High Energy Phys.* **07** (2004) 056 [hep-th/0405245].